Models of computation (MOD) 2015/16

Mid-term exam – March 30, 2016

[Ex. 1]

Let w be the IMP command

$$w \stackrel{\text{def}}{=} \textbf{while } x > y \textbf{ do } (x := x + 1 \ ; \ y := y - 1)$$

1. Characterize the set S of memories σ such that $\langle w, \sigma \rangle \not\rightarrow$ in terms of conditions over $\sigma(x)$ and $\sigma(y)$.

2. Use the inference rule for divergence seen in the course to prove that $\langle w, \sigma \rangle \not\rightarrow$ for any memory $\sigma \in S$.

[Ex. 2]

Let c and c' be the IMP commands defined below:

$$\begin{array}{rcl} c & \stackrel{\text{def}}{=} & \textbf{if } b \textbf{ then } c_1 \textbf{ else } c_2 \\ c' & \stackrel{\text{def}}{=} & (\textbf{if } b \textbf{ then } c_1 \textbf{ else skip}) \ ; \ (\textbf{if } \neg b \textbf{ then } c_2 \textbf{ else skip}) \end{array}$$

Are c and c' equivalent for any $b \in Bexp$ and $c_1, c_2 \in Com$?

1. Motivate the answer by exploiting the denotational semantics.

2. If c and c' are not equivalent provide a concrete counterexample.

[Ex. 3]

Let $(D, \sqsubseteq)$ be a CPO$_\perp$ and $f : D \to D$ be a continuous function on D. We define the set $\mathbf{Po}_f$ of post-fixpoints of f as follows:

$$\mathbf{Po}_f \stackrel{\text{def}}{=} \{ d \in D \mid d \sqsubseteq f(d) \}$$

1. Is $(\mathbf{Po}_f, \sqsubseteq_{\mathbf{Po}_f})$ a CPO$_\perp$, where $\sqsubseteq_{\mathbf{Po}_f} \stackrel{\text{def}}{=} \sqsubseteq \cap (\mathbf{Po}_f \times \mathbf{Po}_f)$?

2. Take $D = \wp(\mathbb{N})$ ordered by inclusion and $f = \lambda S. X \cap S$ for a fixed non-empty subset of natural numbers X. Prove that $\mathbf{Po}_f = \wp(X)$.

[Ex. 4]

Let us consider the signature Σ for binary trees seen during the course, with $\Sigma_0 = \mathbb{N}$, $\Sigma_2 = \{ \mathsf{cons} \}$ and $\Sigma_k = \emptyset$ for all $k \neq 0, 2$.

1. Define by structural recursion the function seq that returns the list of leaves of a tree (such that, e.g., $seq(\mathsf{cons}(\mathsf{cons}(3,1),5)) = 3\ 1\ 5$).

2. Define by structural recursion the function dpt that returns the depth of a tree (such that, e.g., $dpt(\mathsf{cons}(\mathsf{cons}(3,1),5)) = 3$).

3. Let $\asymp \subseteq T_\Sigma \times T_\Sigma$ be the relation defined by the following inference rules:

$$\frac{n \in \mathbb{N}}{n \asymp n} \qquad \frac{n \in \mathbb{N} \quad t_0 \asymp t_1}{\mathsf{cons}(n, t_0) \asymp \mathsf{cons}(n, t_1)} \qquad \frac{t_0 \asymp \mathsf{cons}(n, t_2) \quad \mathsf{cons}(t_2, t_1) \asymp t}{\mathsf{cons}(t_0, t_1) \asymp \mathsf{cons}(n, t)}$$

   Prove $\forall t, t' \in T_\Sigma.\ t \asymp t' \ \Rightarrow \ (\, seq(t) = seq(t') \wedge dpt(t') = |seq(t)|\,)$, where $|s|$ denotes the length of the sequence s. Is the converse true?