# Methods for the specification and verification of business processes

## MPB (6 cfu, 295AA)

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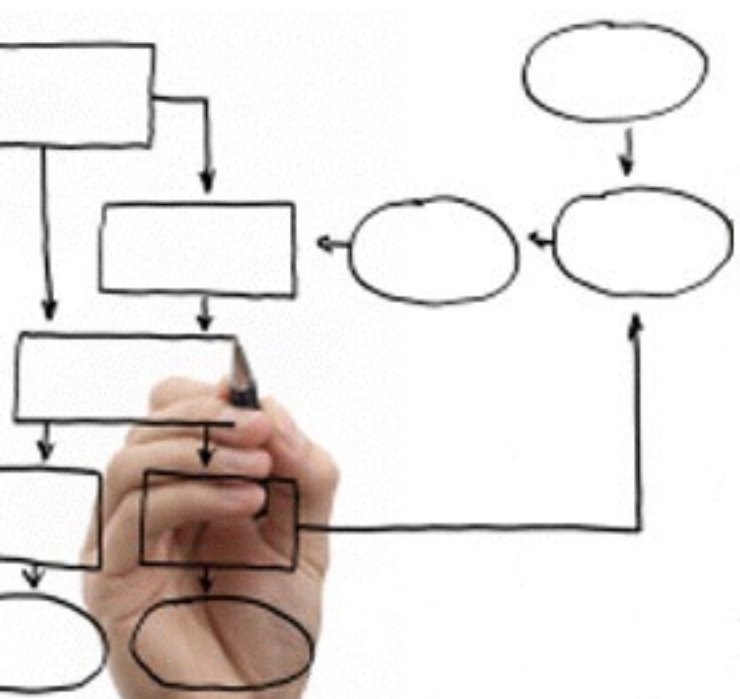
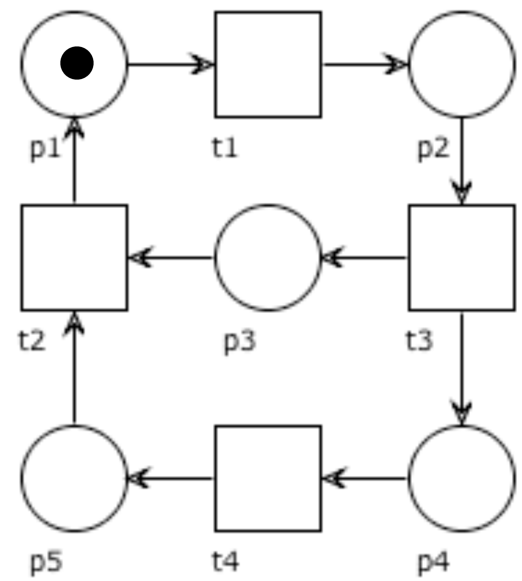


## 17 - T-systems

# Object




We study some "good" properties of T-systems

Free Choice Nets (book, optional reading)
https://www7.in.tum.de/~esparza/bookfc.html

# T-systems

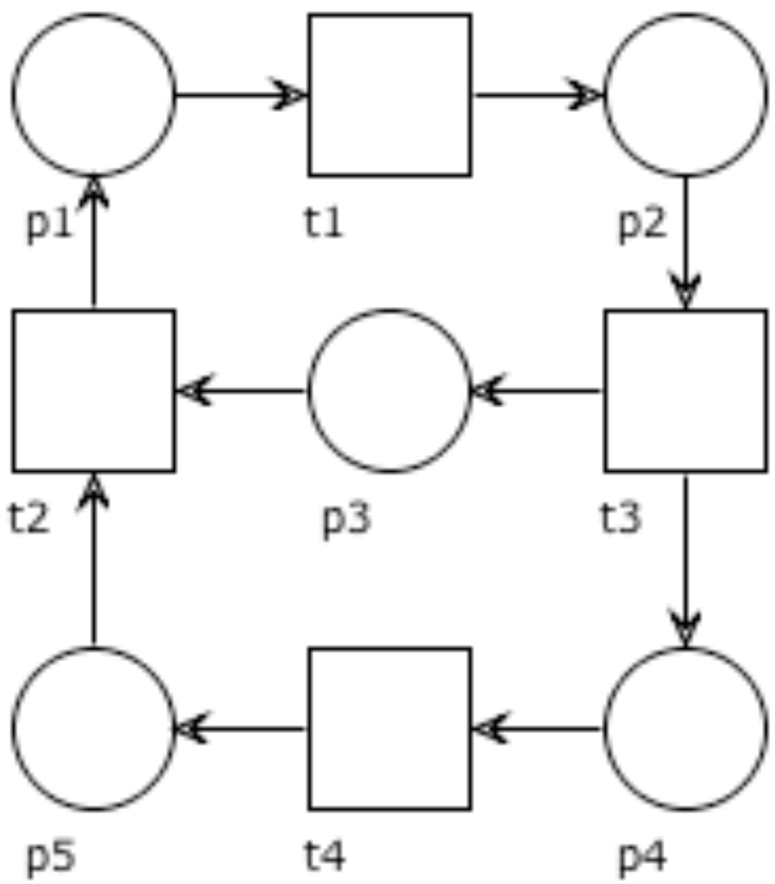
# T-system

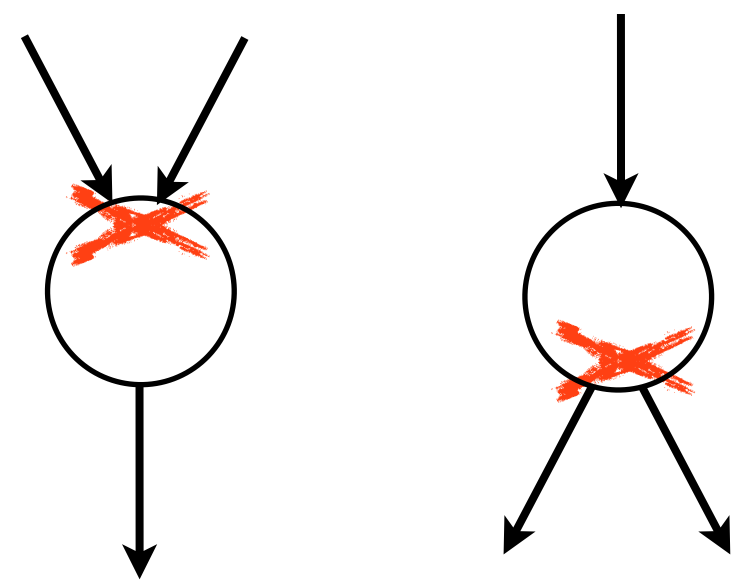
**Definition**: We recall that a net N is a **T-net** if each place has exactly one input transition and exactly one output transition

$$\forall p \in P, \qquad |\bullet p| = 1 = |p \bullet|$$

A system $(N,M_0)$ is a **T-system** if N is a T-net

# T-net: example



# T-systems: an observation

Notably, computation in T-systems is concurrent, but essentially deterministic:

the firing of a transition t in M cannot disable another transition t' enabled at M

# T-net N*

Is the following conjecture true?
A workflow net N is a T-net
iff N* is a T-net

# T-net N*

Is the following conjecture true?
A workflow net N is a T-net
iff N* is a T-net

No, a workflow net cannot be a T-net because
the place i has no incoming arc
and the place o has no outgoing arc

(N* can be a T-net)

# T-systems: another observation

Determination of control:

the transitions responsible for enabling t are one for each input place of t

# Notation: token count of a circuit

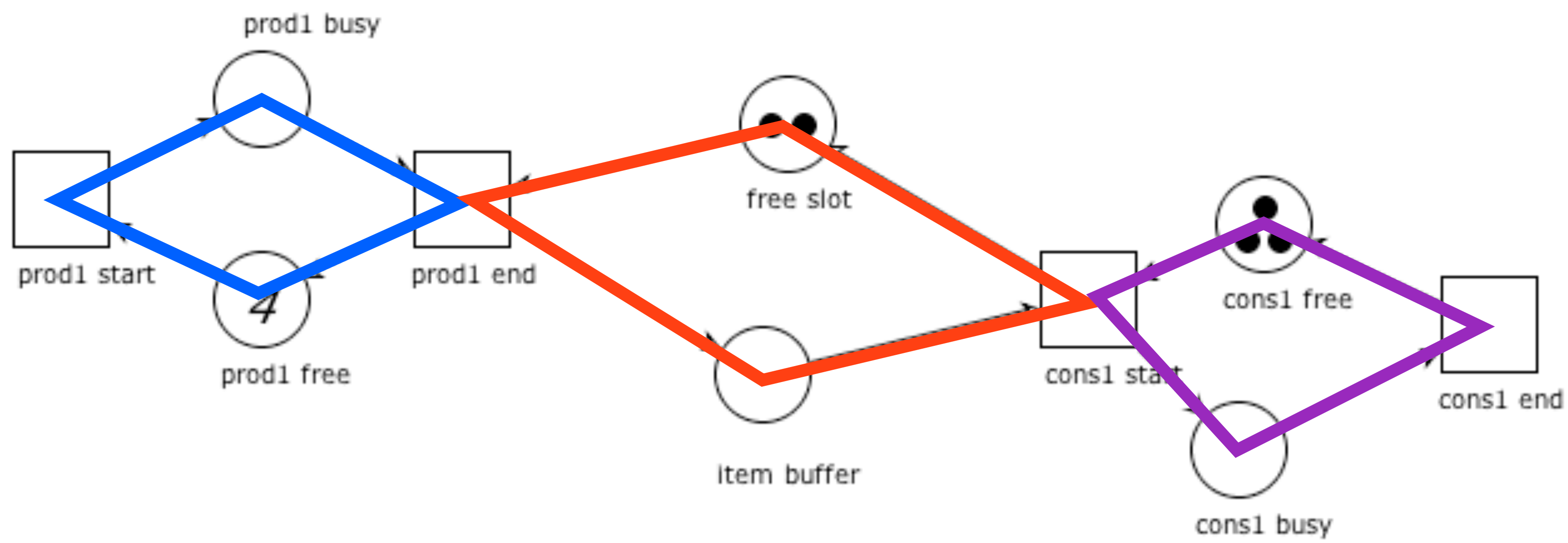
Let $\gamma = (x_1, y_1)(y_1, x_2)(x_2, y_2)...(x_n, y_n)$ be a circuit.

Let $P_{|\gamma} \subseteq P$ be the set of places in $\gamma$.

$$M(\gamma) = M(P_{|\gamma}) = \sum_{p \in P_{|\gamma}} M(p)$$

We say that $\gamma$ is **marked at** $M$ if $M(\gamma) > 0$

# Example



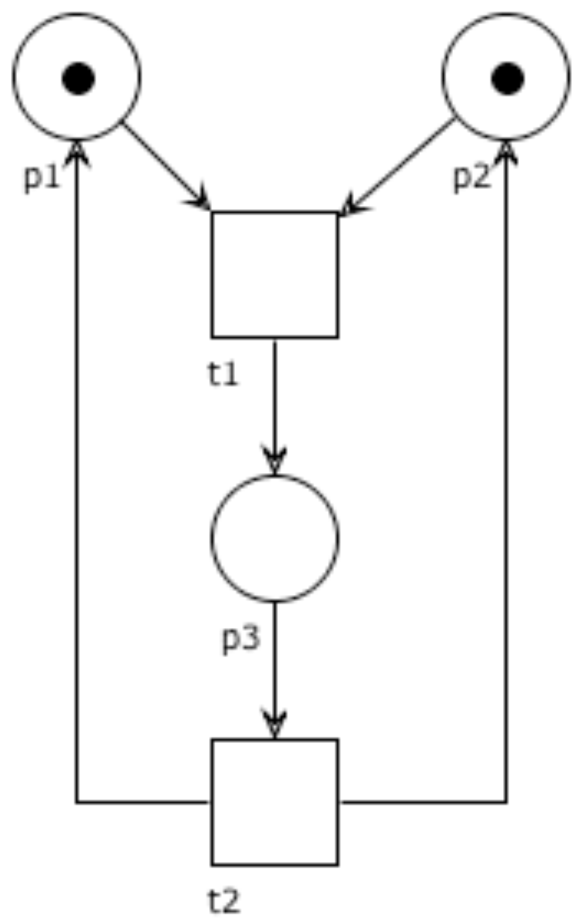

$$M(\gamma_1) = 4$$

$$M(\gamma_2) = 2$$

$$M(\gamma_3) = 3$$

# Question time

Trace two circuits over the T-system below



# Fundamental property of T-systems

The token count of a circuit is invariant under any firing.

# Fundamental property of T-systems

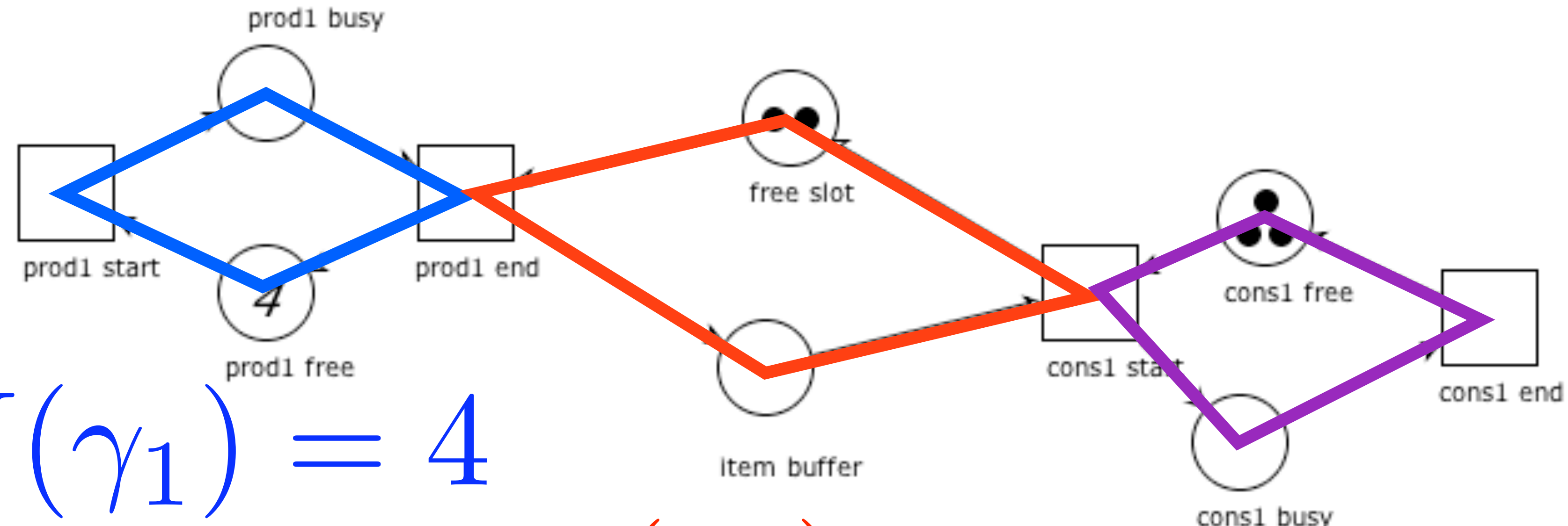
**Proposition**: Let $\gamma$ be a circuit of a T-system $(P, T, F, M_0)$. If $M$ is a reachable marking, then $M(\gamma) = M_0(\gamma)$

Take any $t \in T$: either $t \notin \gamma$ or $t \in \gamma$.

If $t \notin \gamma$, then no place in $\bullet t \cup t \bullet$ is in $\gamma$ (otherwise, by definition of T-nets, $t$ would be in $\gamma$). Then, an occurrence of $t$ does not change the token count of $\gamma$.

If $t \in \gamma$, then exactly one place in $\bullet t$ and one place in $t \bullet$ are in $\gamma$. Then, an occurrence of $t$ does not change the token count of $\gamma$.

# Example




$$M(\gamma_1) = 4$$

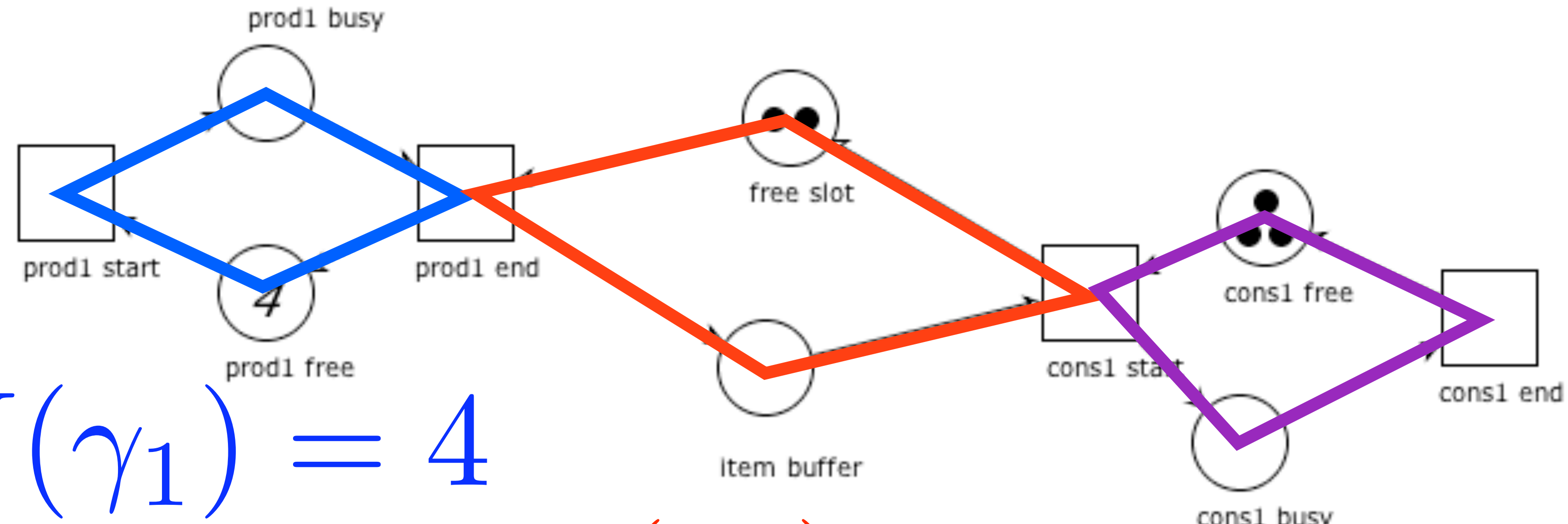
$$M(\gamma_2) = 2$$

$$M(\gamma_3) = 3$$

$M_0$ = [ 0 4 2 0 3 0 ]

M = [ 2 2 1 2 2 1 ]

# Example




$$M(\gamma_1) = 4$$

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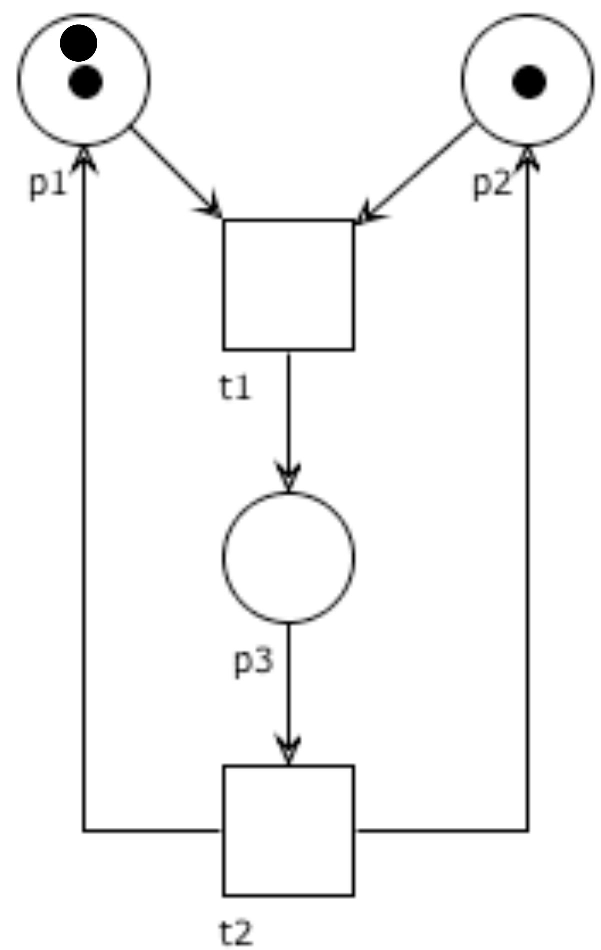
$M_0$ = [ 0 4 2 0 3 0 ]

M’ = [ 2 1 1 1 2 2 ]

# Question time

Is the marking $p_1 + 2p_2$ reachable? (why?)



# T-invariants of T-nets

**Proposition**: Let N=(P,T,F) be a (connected) T-net. **J** is a T-invariant of N **iff** **J**=[ k ... k ] for some value k

(the proof is dual to the analogous proposition for S-invariants of S-nets)

# Boundedness in strongly connected T-systems

**Lemma**: If a T-system $(N,M_0)$ is strongly connected, then it is bounded

Let $\Gamma$ be the set of the circuits of $N$ and let $k = \max_{\gamma \in \Gamma} M_0(\gamma)$.

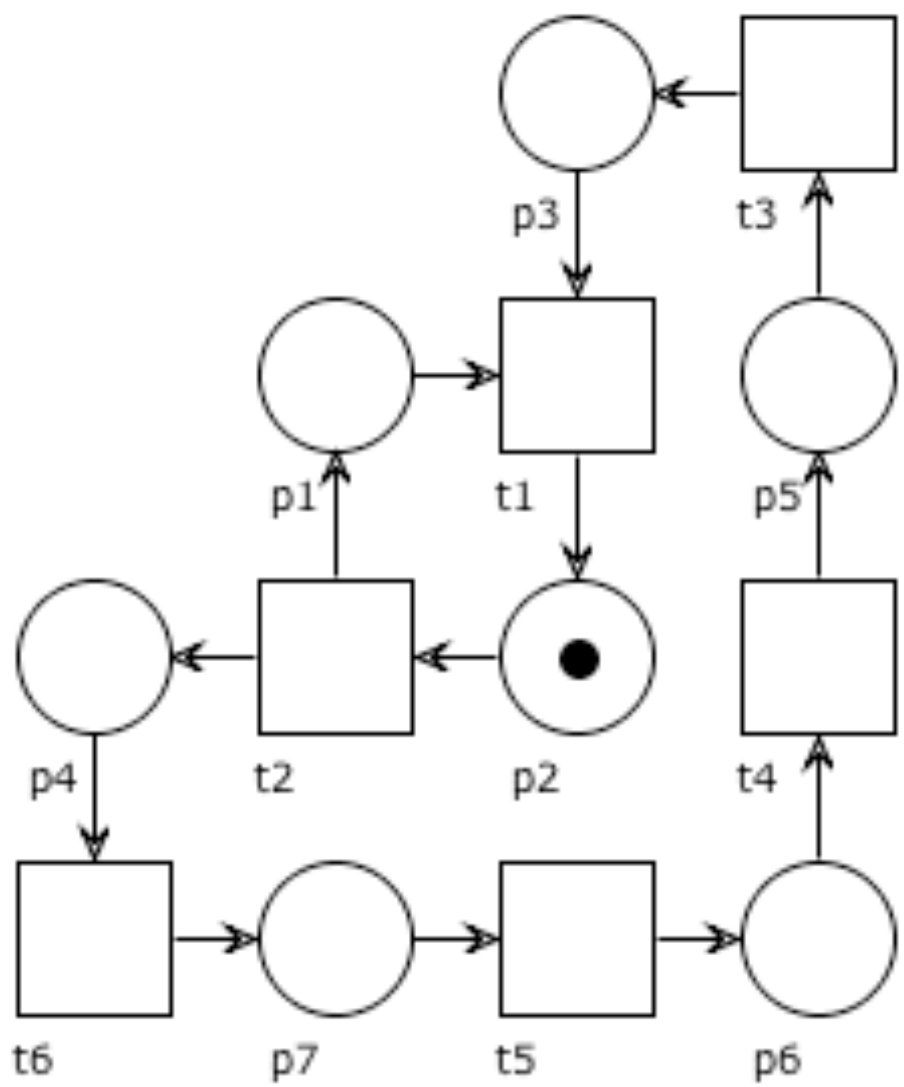
Since $N$ is strongly connected, every place $p$ belongs to some circuit $\gamma_p$.

By the fundamental property of T-systems: token count of $\gamma_p$ is invariant.

Thus, for any reachable marking $M$, we have $M(p) \leq M(\gamma_p) = M_0(\gamma_p) \leq k$.
Hence the net is $k$-bounded.

# Question time

Is the T-systems below bounded? (why?)



# Liveness theorem for T-systems

**Theorem**: A T-system (N,$M_0$) is live
**iff** every circuit of N is marked at $M_0$

$\Rightarrow$) (quite obvious)
By contradiction, let $\gamma$ be a circuit with $M_0(\gamma) = 0$.
By the fundamental property of T-systems: $\forall M \in [\, M_0 \,\rangle,\ M(\gamma) = 0$.

Take any $t \in T_{|\gamma}$ and $p \in P_{|\gamma} \cap \bullet t$.

For any $M \in [\, M_0 \,\rangle$, we have $M(p) = 0$.
Hence $t$ is never enabled and the T-system is not live.

# Liveness theorem for T-systems

**Theorem**: A T-system (N,$M_0$) is live
**iff** every circuit of N is marked at $M_0$

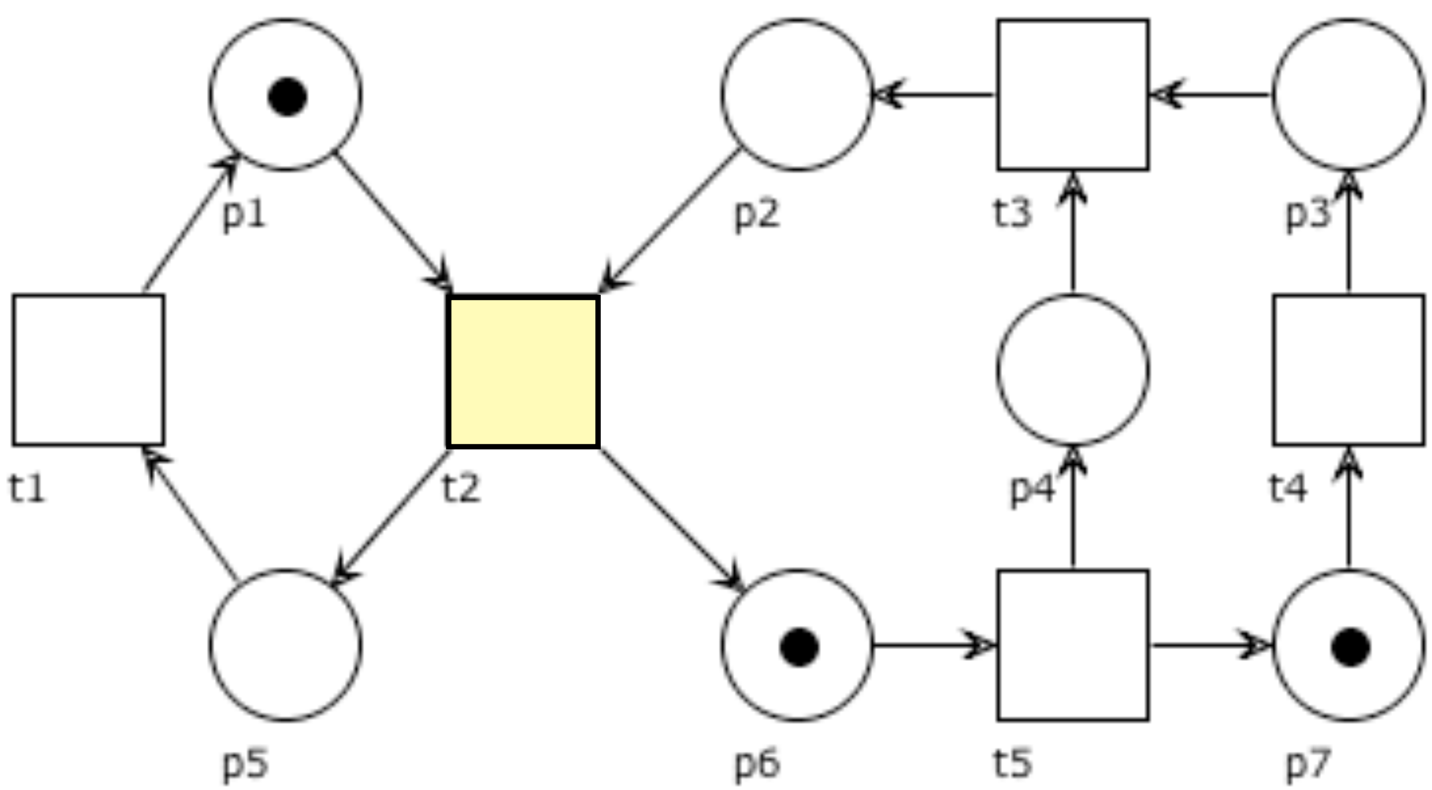
$\Leftarrow$) (more involved)
Take any $t \in T$ and $M \in [\, M_0 \,\rangle$.
We need to show that some marking $M'$ reachable from $M$ enables $t$.

The key idea is to collect the places that control the firing of $t$:
$p \in P_{M,t}$ if there is a path from $p$ to $t$ through places unmarked at $M$.
We then proceed by induction on the size of $P_{M,t}$.

We just sketch the key idea of the proof over a T-system.

# Liveness theorem for T-systems

**Theorem**: A T-system $(N,M_0)$ is live
$\Leftarrow$ every circuit of N is marked at $M_0$




$M = p_1 + p_6 + p_7$

M' enabling $t_2$?

# Liveness theorem for T-systems

**Theorem**: A T-system $(N,M_0)$ is live
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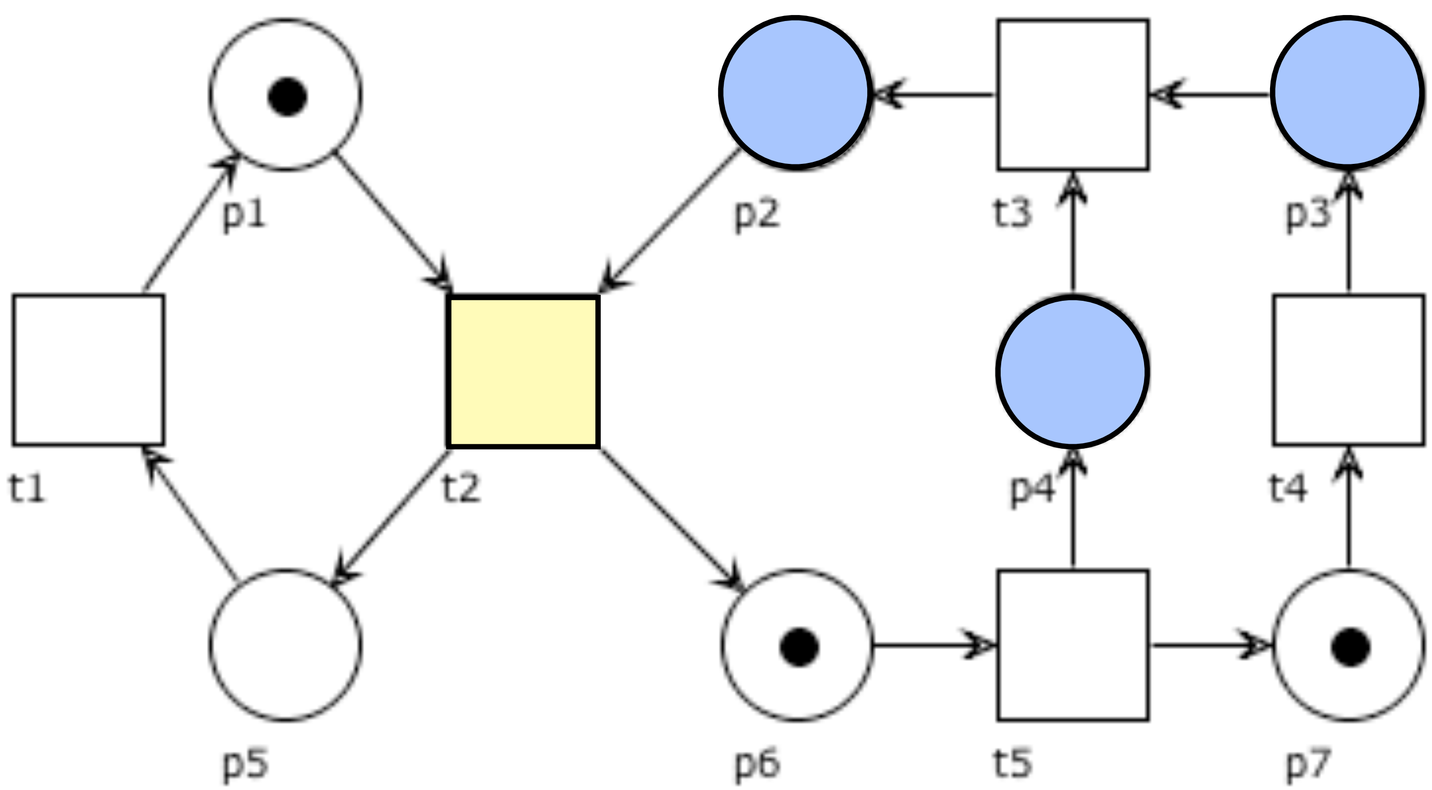



$P_{M,t_2} = \{ p_2, p_3, p_4 \}$

# Liveness theorem for T-systems

**Theorem**: A T-system $(N, M_0)$ is live
$\Leftarrow$ every circuit of N is marked at $M_0$

$\Leftarrow$) (continued proof sketch)

Base case: $|P_{M,t}| = 0$.

Every place in $\bullet t$ is already marked at $M$.

Hence $t$ is enabled at $M$.

# Liveness theorem for T-systems

**Theorem**: A T-system $(N,M_0)$ is live
$\Leftarrow$ every circuit of N is marked at $M_0$

$\Leftarrow$) (continued proof sketch)

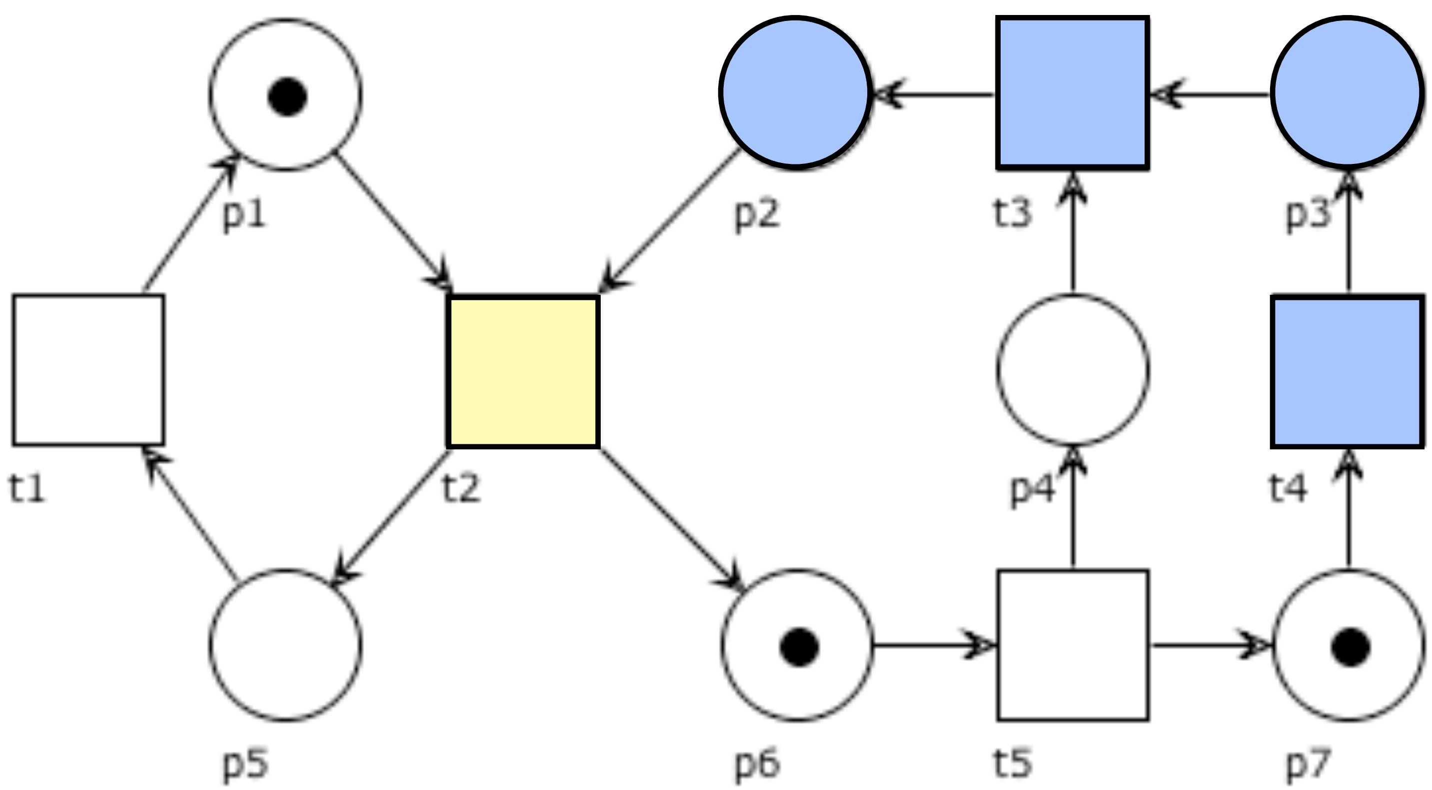
Inductive case: $|P_{M,t}| > 0$.
Therefore $t$ is not enabled at $M$.

We look for a path $\pi$ of maximal length necessary for firing $t$.
$\pi$ must contain only places unmarked at $M$.

By the fundamental property of T-systems: all circuits are marked at $M$.
$\pi$ is not necessarily unique, but exists (no cycle in it).

# Liveness theorem for T-systems

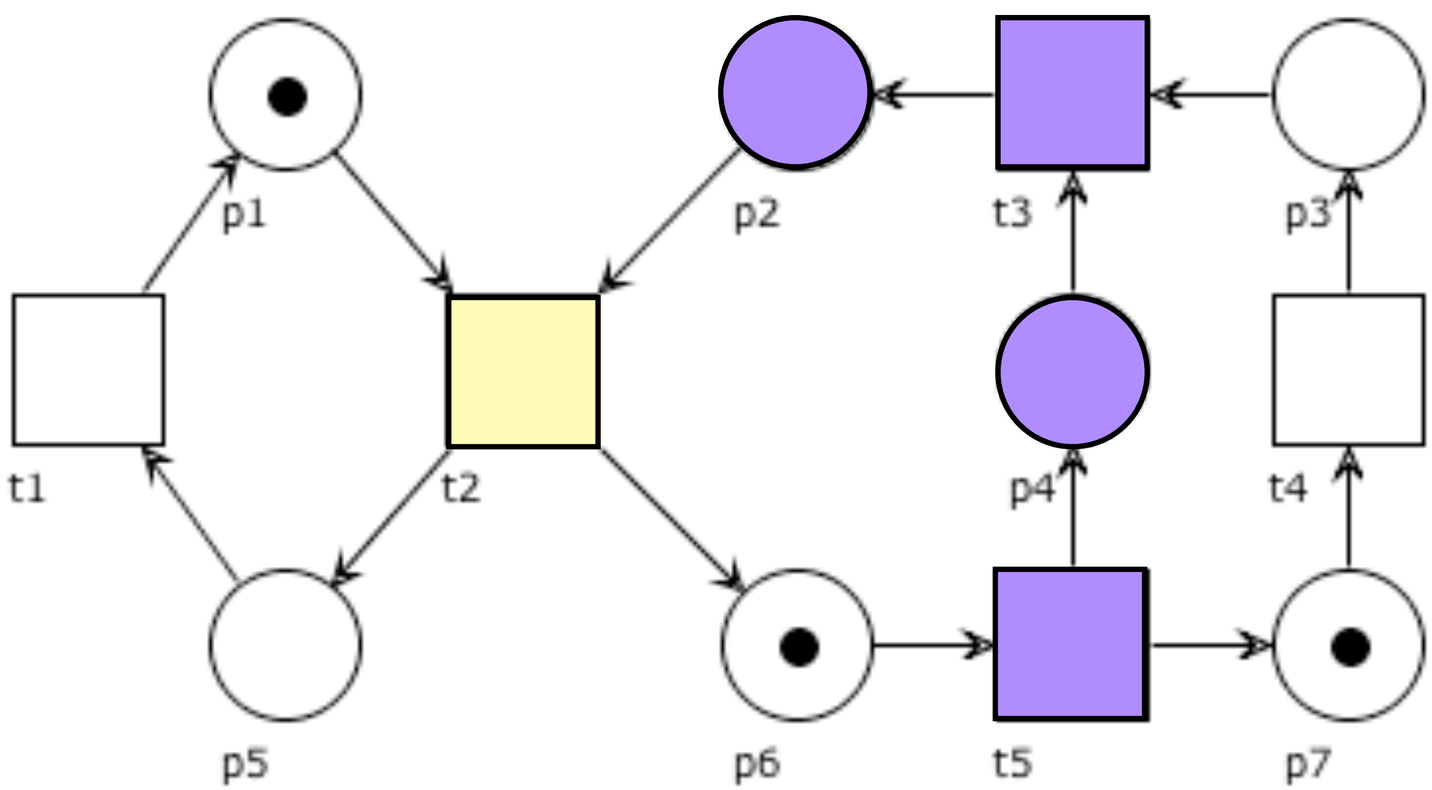
**Theorem**: A T-system $(N,M_0)$ is live $\Leftarrow$ every circuit of N is marked at $M_0$




$$\pi = t_4\, p_3\, t_3\, p_2\, t_2$$

# Liveness theorem for T-systems

**Theorem**: A T-system $(N,M_0)$ is live
$\Leftarrow$ every circuit of N is marked at $M_0$




$$\pi = t_5\, p_4\, t_3\, p_2\, t_2$$

# Liveness theorem for T-systems

**Theorem**: A T-system $(N,M_0)$ is live
$\Leftarrow$ every circuit of N is marked at $M_0$

$\Leftarrow$) (Inductive case: $|P_{M,t}| > 0$, continued proof sketch)

$\pi$ begins with a transition $t'$ enabled at $M$.
(otherwise a longer path could be found).

By firing $t'$ we reach a marking $M''$ such that $P_{M'',t} \subset P_{M,t}$.

Hence $|P_{M'',t}| < |P_{M,t}|$ and we conclude by inductive hypothesis.

# Liveness theorem for T-systems

**Theorem**: A T-system $(N,M_0)$ is live
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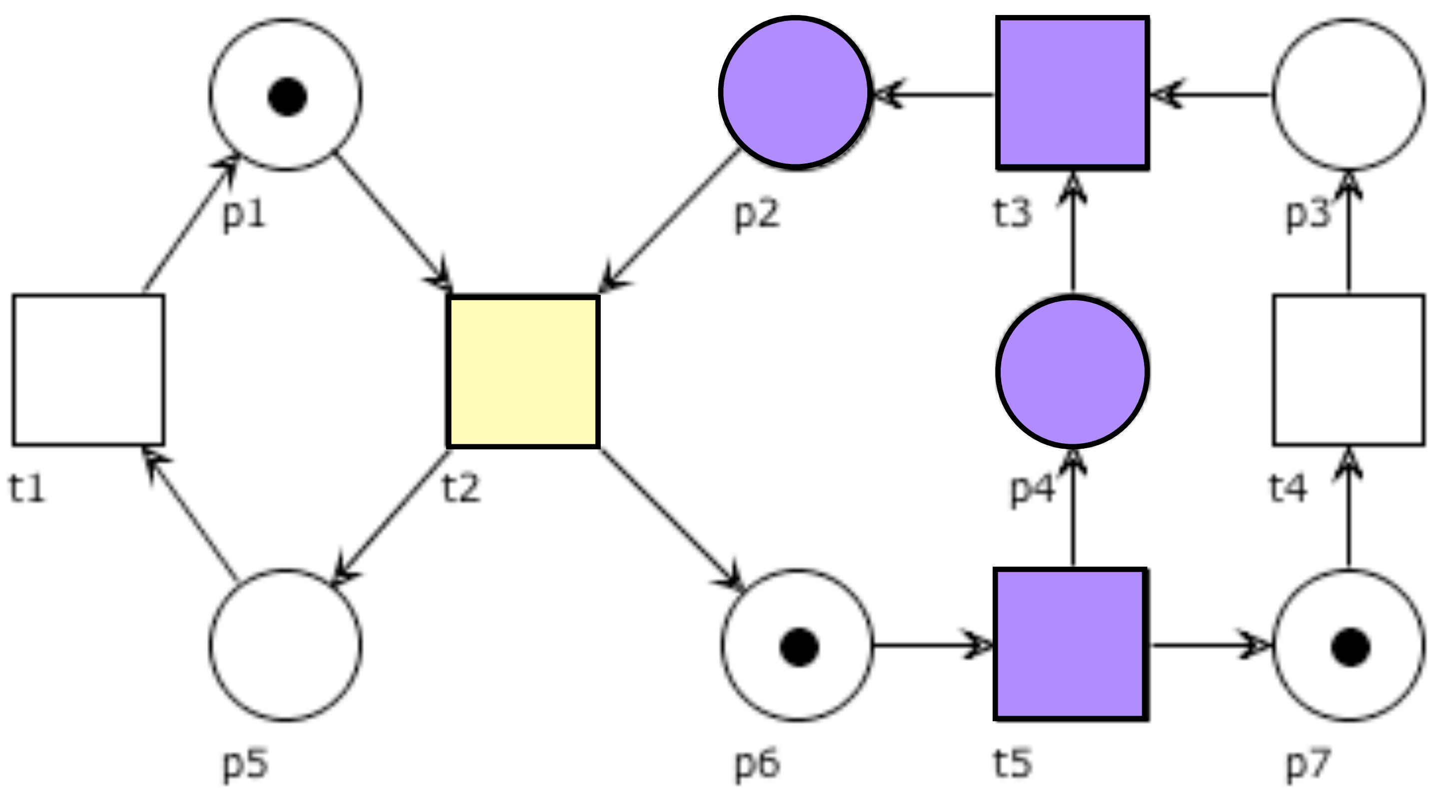


$P_{M,t_2} = \{ p_2, p_3, p_4 \}$

$$\pi = t_5 \, p_4 \, t_3 \, p_2 \, t_2$$

# Liveness theorem for T-systems

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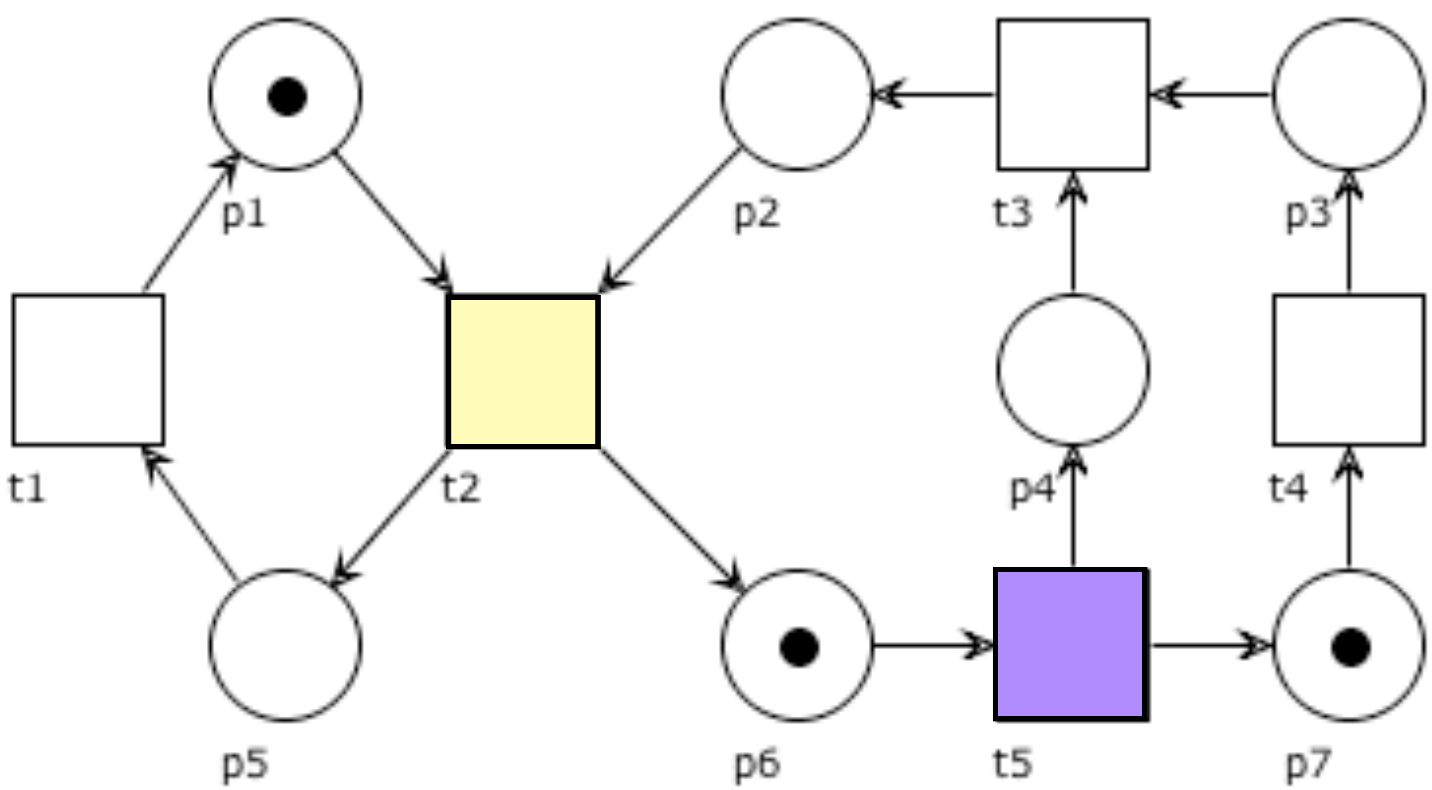


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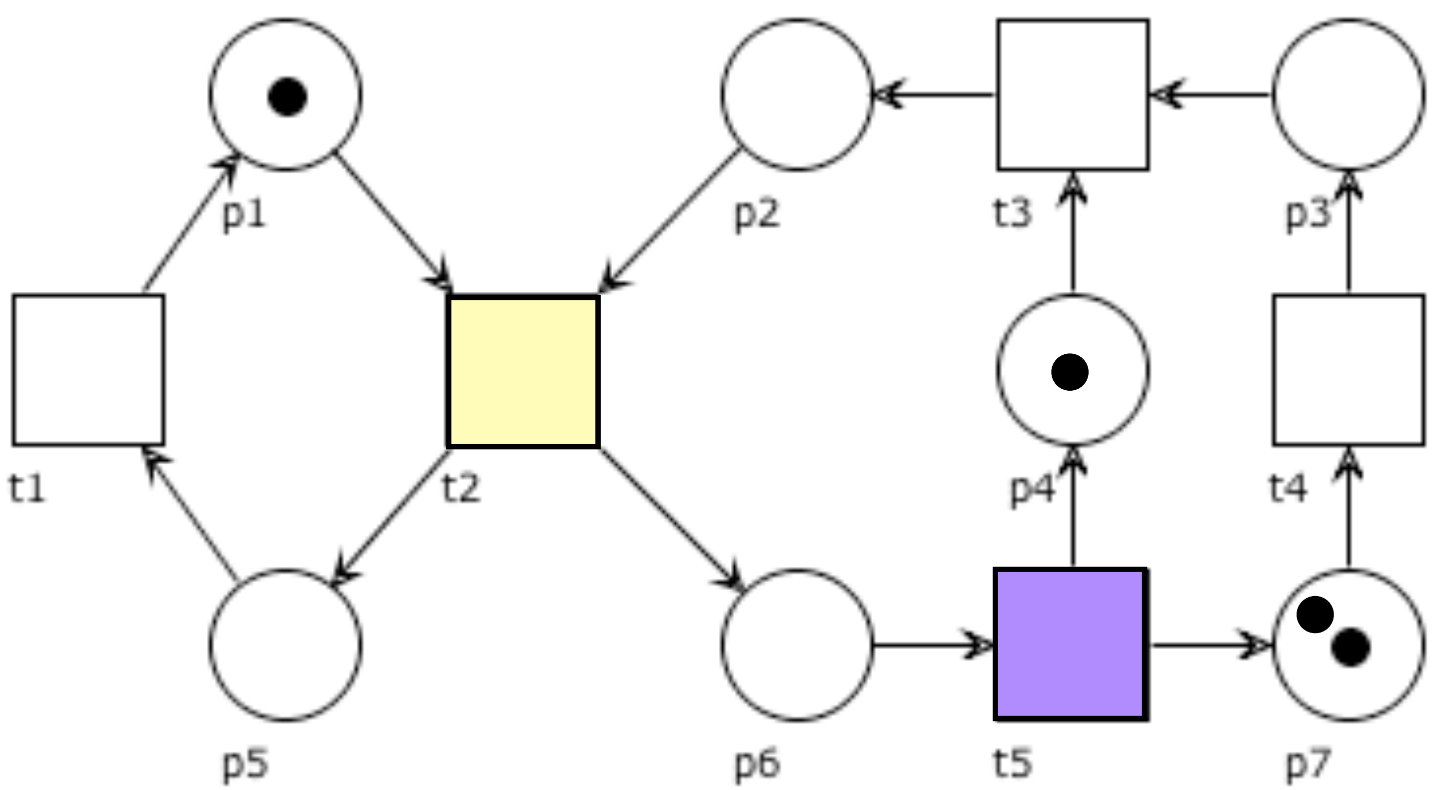


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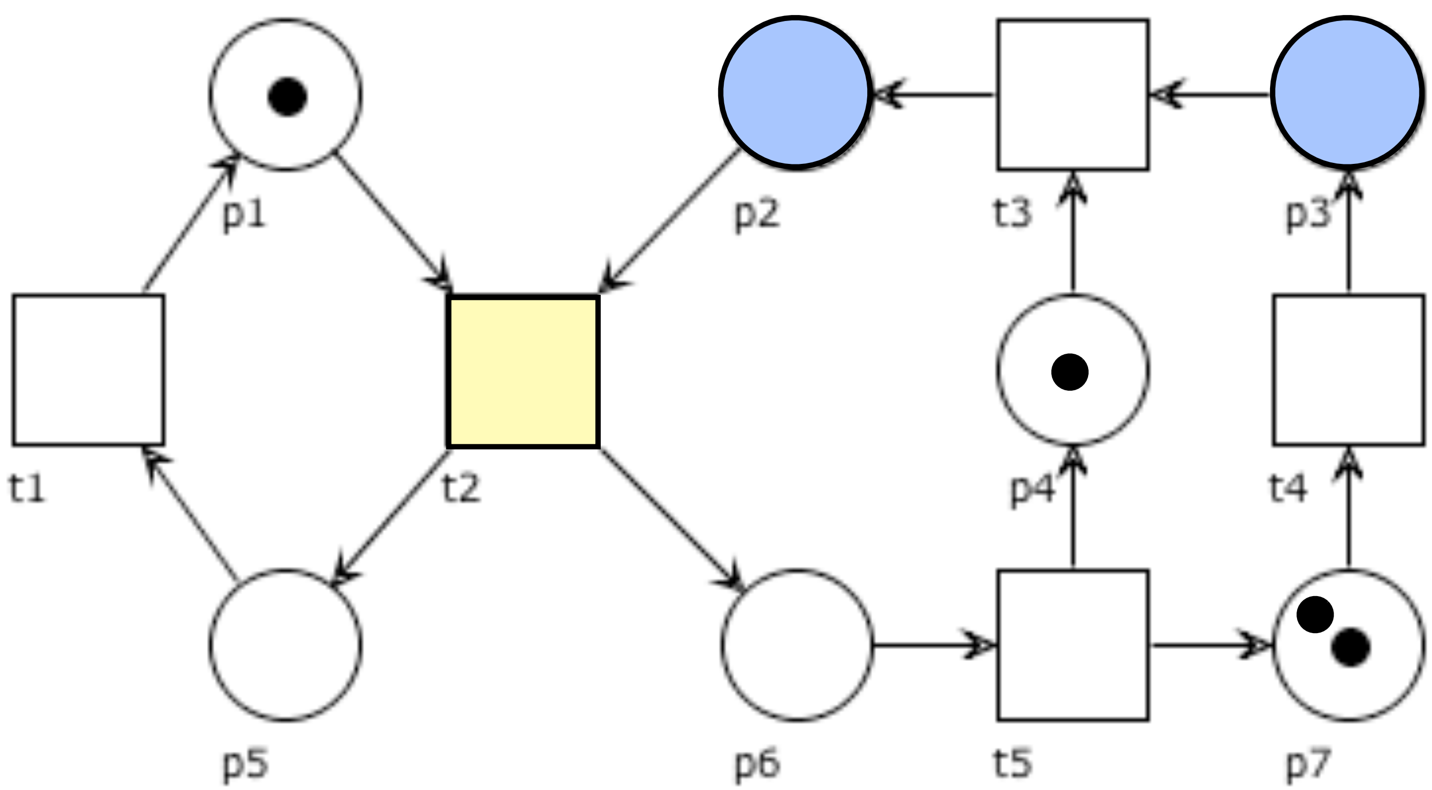
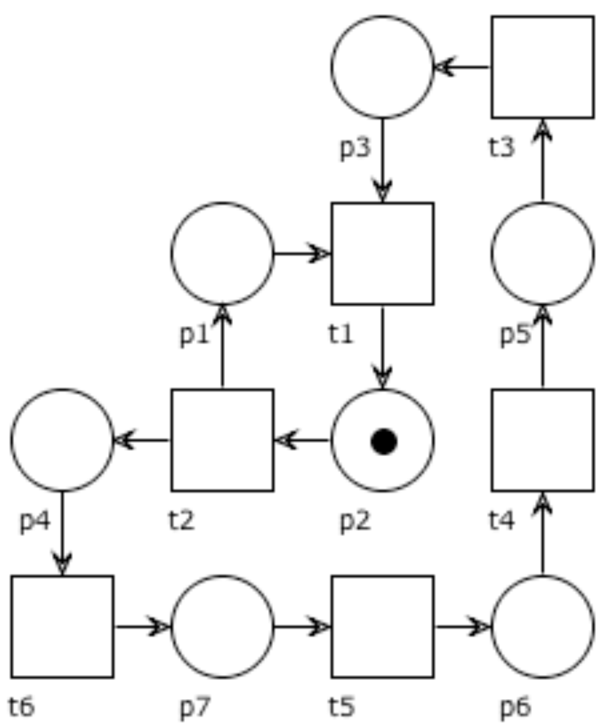


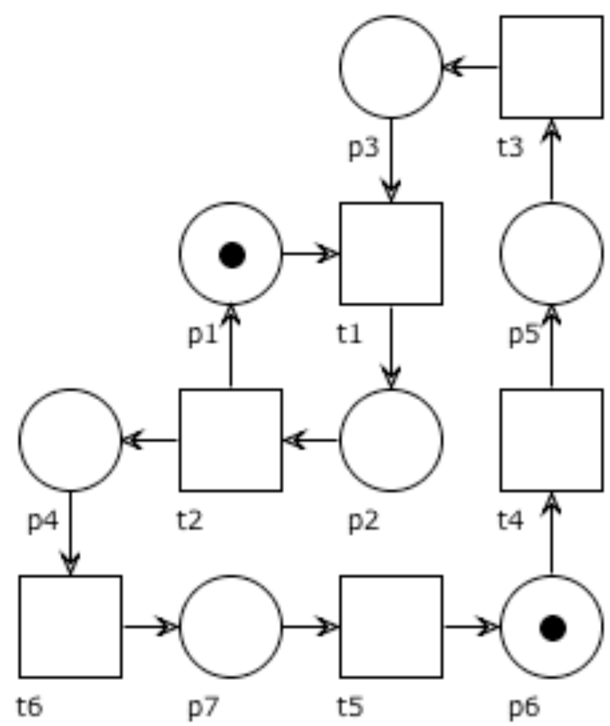

$P_{M,t_2} = \{ p_2, p_3, p_4 \}$

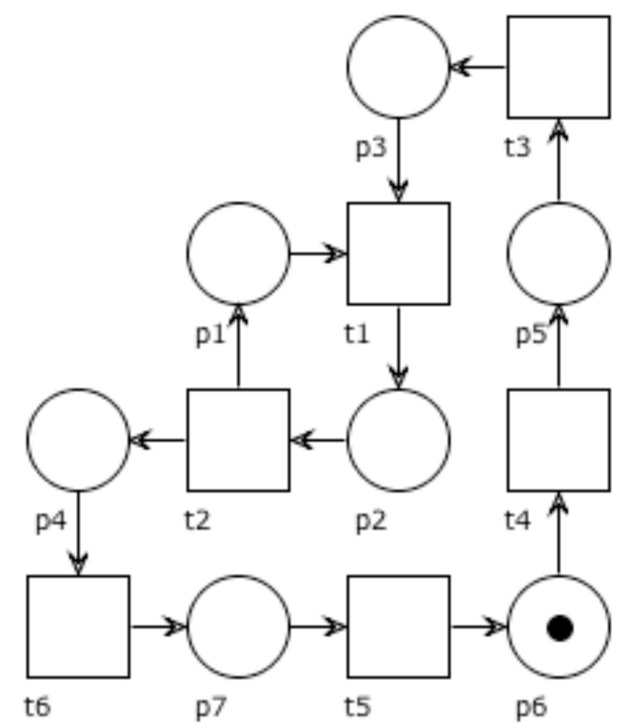
$P_{M'',t_2} = \{ p_2, p_3 \}$

# Question time

Which of the T-systems below is live? (why?)



# Boundedness theorem for live T-systems

**Theorem**: A live T-system $(P, T, F, M_0)$ is $k$-bounded iff every place $p \in P$ belongs to a circuit $\gamma_p$ with $M_0(\gamma_p) \leq k$.

$\Leftarrow$) Let $M \in [\, M_0 \,\rangle$ and take any $p \in P$.

By the fundamental property of T-systems:
$M(p) \leq M(\gamma_p) = M_0(\gamma_p) \leq k$

# Boundedness theorem for live T-systems

**Theorem**: A live T-system $(P, T, F, M_0)$ is $k$-bounded iff every place $p \in P$ belongs to a circuit $\gamma_p$ with $M_0(\gamma_p) \leq k$.

$\Rightarrow$) Let $k_p \leq k$ be the bound of $p$.
Take $M \in [\, M_0 \,\rangle$ with $M(p) = k_p$.

Define $L = M - k_p p$ and note that the T-system $(N, L)$ is not live.
(otherwise $L \xrightarrow{\sigma} L'$ with $L'(p) > 0$ for enabling $t \in p\bullet$. But then:
$M = L + k_p p \xrightarrow{\sigma} L' + k_p p = M'$ with $M'(p) = L'(p) + k_p > k_p$!)

By the liveness theorem: some circuit $\gamma$ is not marked at $L$.
Since $(N, M)$ is live, the circuit $\gamma$ is marked at $M \supset L$.
Since $M - L = k_p p$, the circuit $\gamma$ contains $p$ and
$M_0(\gamma) = M(\gamma) = M(p) = k_p \leq k$.

# Place bounds in live T-systems

Let $(P, T, F, M_0)$ be a **live** T-system.
We can draw some easy consequences of the above results:

1) If $p \in P$ is bounded, then it belongs to some circuit.
(see part $\Rightarrow$ of the proof of the boundedness theorem)

2) If $p \in P$ belongs to some circuit, then it is bounded.
(by the fundamental property of T-systems)

3) If $(N, M_0)$ is bounded, then it is strongly connected.
(by strong connectedness theorem, holding for any system)

4) If $N$ is strongly connected, then $(N, M_0)$ is bounded.
(by 1, since any $p \in P$ belongs to a circuit by strong connectdness)

# Place bounds in live T-systems

Let $(P, T, F, M_0)$ be a **live** T-system.
We can draw some easy consequences of the above results:

1+2) $p \in P$ is bounded iff it belongs to some circuit.

3+4) $(N, M_0)$ is bounded iff it is strongly connected.

# T-systems: recap

T-system + $\gamma$ circuit + M reachable => $M(\gamma) = M_0(\gamma)$

T-system + $\gamma$ circuit + $M(\gamma) \neq M_0(\gamma)$ => M not reachable

T-system + $\gamma_1 \ldots \gamma_n$ circuits: $\exists i.\ p \in \gamma_i$ <=> p bounded

T-system: $M_0(\gamma) > 0$ for all circuits $\gamma$ <=> live

T-system: strongly connected => bounded

T-system + live: strongly connected <=> bounded

T-system: T-invariant **J** <=> **J** = [ k k ... k ]

# Consequences on workflow nets

**Theorem**: If N is a workflow net s.t. N* is a T-system then N is safe and sound **iff** every circuit of N* is marked

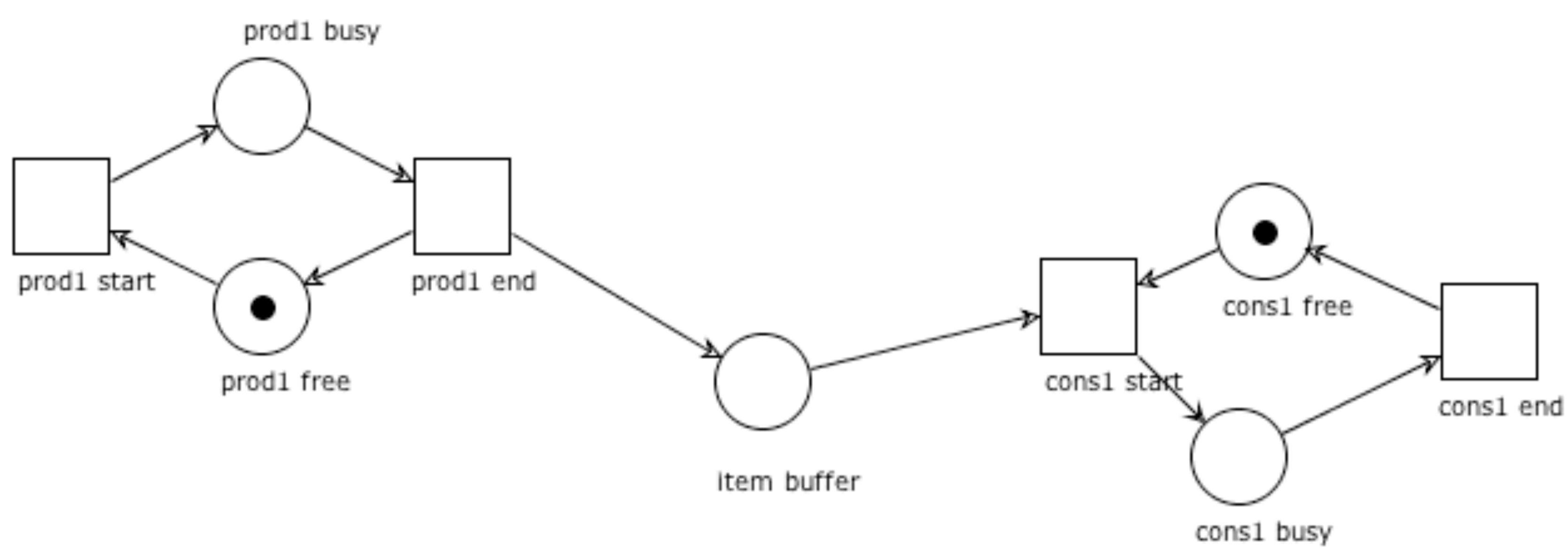
N workflow net => N* strongly connected
all circuits of N* are marked <=> N* live
N* strongly connected + N* T-system => N* bounded
i $\in \gamma$ <=> $M_0(\gamma)=1$ <=> $\gamma$ marked circuit

$\gamma$ marked circuit + M reachable => $M(\gamma)=1$

N* bounded <=> any place p belongs to a circuit of N*
p belongs to a circuit of N* => p is safe
all places belong to marked circuits => N* safe => N safe

# Exercises

Which are the circuits of the T-system below?
Is the T-system below live? (why?)
Which places are bounded? (why?)
Assign a bound to each bounded place.



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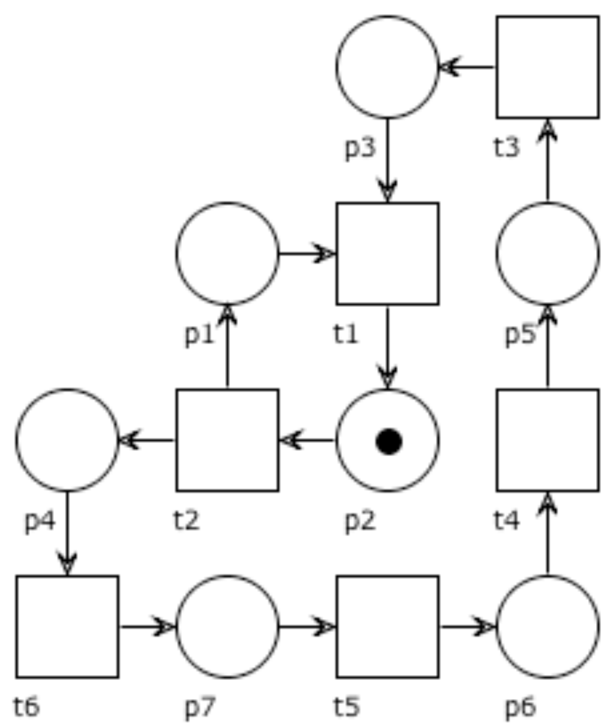
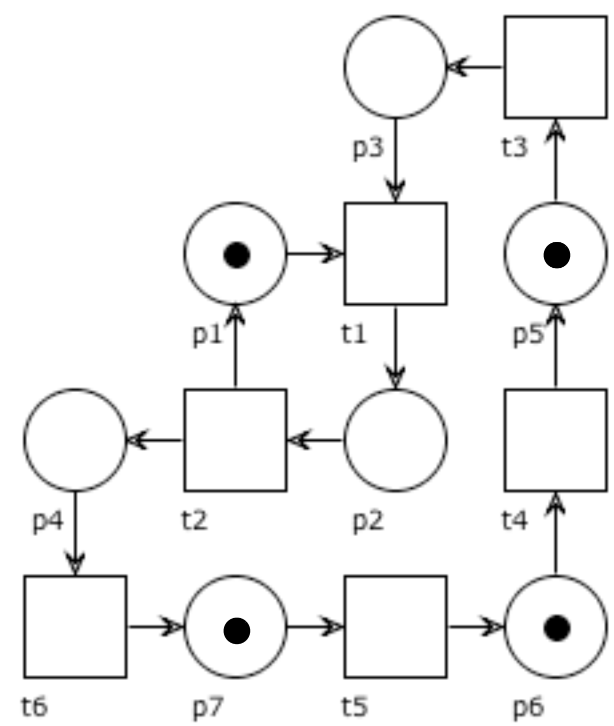
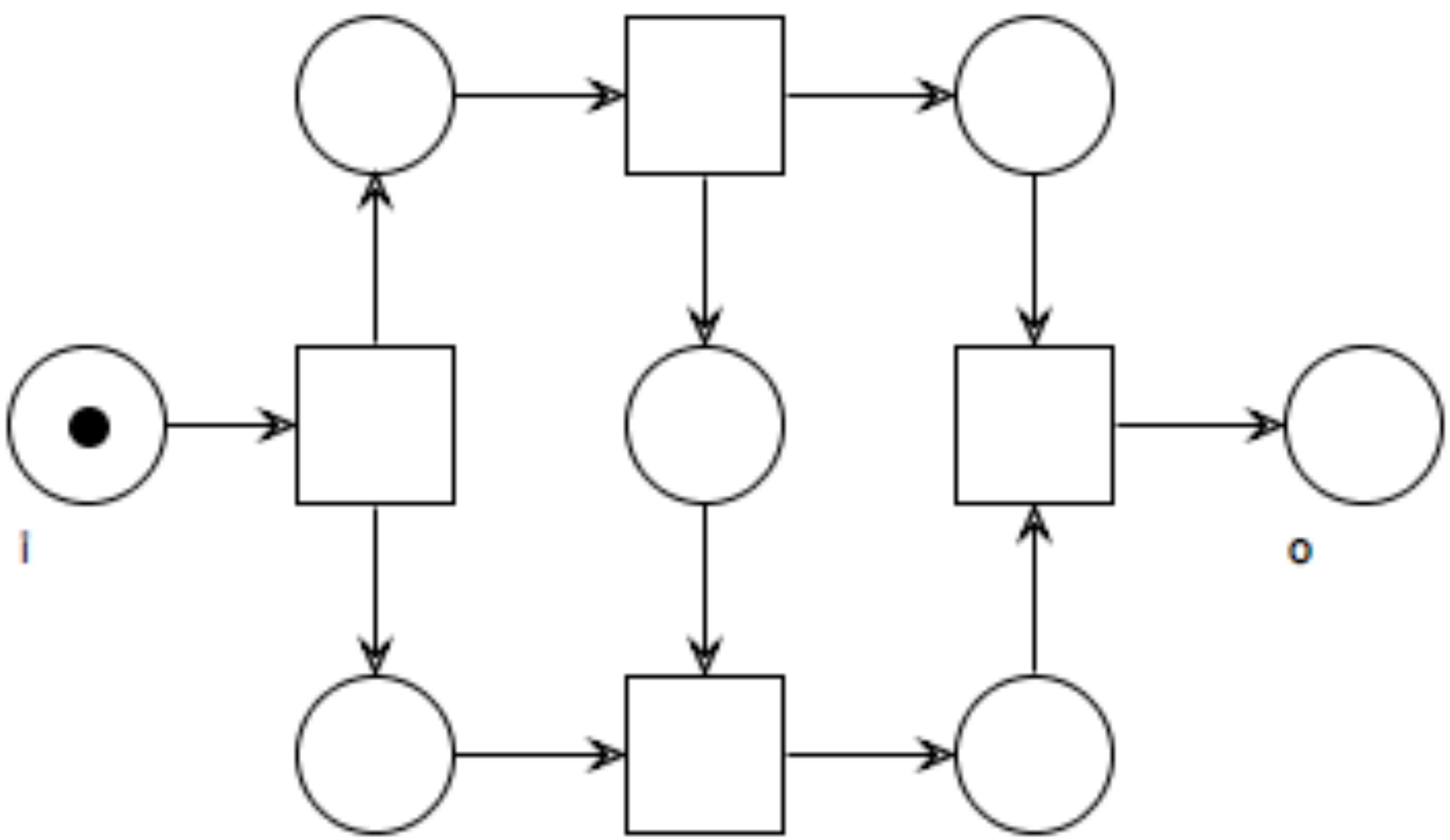


# Exercise

Is the net below a workflow net?
Is it sound?



# Exercise

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