Master Program in *Data Science and Business Informatics*

# Statistics for Data Science

## Lesson 08 - Continuous random variables


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# Discrete random variables

DEFINITION. Let $\Omega$ be a sample space. A *discrete random variable* is a function $X : \Omega \to \mathbb{R}$ that takes on a finite number of values $a_1, a_2, \ldots, a_n$ or an infinite number of values $a_1, a_2, \ldots$.

DEFINITION. The *probability mass function* $p$ of a discrete random variable $X$ is the function $p : \mathbb{R} \to [0, 1]$, defined by

$$p(a) = \mathrm{P}(X = a) \qquad \text{for } -\infty < a < \infty.$$

- Support finite or countable $\{a_1, \ldots, a_n, \ldots\}$
  - $p(a_i) > 0$ for $i = 1, 2, \ldots$
  - $p(a) = 0$ if $a \notin \{a_1, a_2, \ldots\}$
  - $\sum_i p(a_i) = 1$
- What happens when the support is uncountable? E.g., $[0, 1]$ or $\mathbb{R}^+$ or $\mathbb{R}$
  - Many observations belong to the continuum (time, height, weight, blood pressure, temperature, distance, speed, etc.)

# Discrete random variables

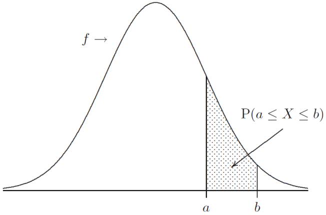
- Le $X$ with support $\{0, 1\}$
  - $p(a) = P(X = a) = 1/2$ for $a$ in the support
- Assume to expand the support to $\{0, 1/n, 2/n, \ldots, n/n{-}1, 1\}$
  - $p(a) = P(X = a) = 1/(n+1)$ for $a$ in the support
- Ok for $n \in \mathbb{N}$, but for $n \to \infty$, we have:

$$p(a) = P(X = a) = 0 \quad \text{for all } a$$

  which break the requirements of distribution function!
- Since $|\mathbb{R}| = 2^{\aleph_0} > \aleph_0 = |\mathbb{N}|$, $n = \infty$ is reached when considering the continuum!
- **Conclusion:** the idea of probability mass function does not extend to the continuum!

# Continuous random variables

- We cannot assign a "mass" to a real number, but we can assign it to an interval!




DEFINITION. A random variable $X$ is *continuous* if for some function $f : \mathbb{R} \to \mathbb{R}$ and for any numbers $a$ and $b$ with $a \le b$,

$$\mathrm{P}(a \le X \le b) = \int_a^b f(x)\,\mathrm{d}x.$$

The function $f$ has to satisfy $f(x) \ge 0$ for all $x$ and $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$. We call $f$ the *probability density function* (or *probability density*) of $X$.

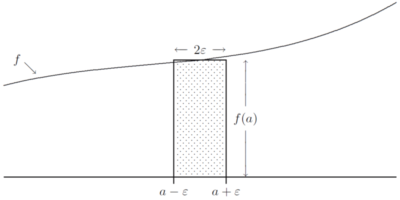
- Support of $X$ is $\{x \in \mathbb{R} \mid f(x) > 0\}$
- $F(a) = P(X \le a) = \int_{-\infty}^{a} f(x)dx$ *[Cumulative Distribution Function]*

# Density function

$$P(X = a) \leq P(a - \epsilon \leq X \leq a + \epsilon) = \int_{a-\epsilon}^{a+\epsilon} f(x)dx = F(a + \epsilon) - F(a - \epsilon)$$

- for $\epsilon \to 0$, $P(a - \epsilon \leq X \leq a + \epsilon) \to 0$, hence $P(X = a) = 0$

- What is the meaning of the density function $f(x)$?
  - $f(a)$ is a (relative) measure of how likely is $X$ will be near $a$
  - "probability mass per unit length" around $a$: $f(a) \cdot 2\epsilon$




- Discrete vs Continuous Random Variables

$$F(a) = \sum_{a_i \leq a} p(a_i) \quad p(a_i) = F(a_i) - F(a_{i-1}) \qquad F(x) = \int_{-\infty}^{x} f(y)dy \quad f(x) = \frac{d}{dx}F(x)$$

# $X \sim U(\alpha, \beta)$

DEFINITION. A continuous random variable has a *uniform distribution* on the interval $[\alpha, \beta]$ if its probability density function $f$ is given by $f(x) = 0$ if $x$ is not in $[\alpha, \beta]$ and

$$f(x) = \frac{1}{\beta - \alpha} \quad \text{for } \alpha \leq x \leq \beta.$$

We denote this distribution by $U(\alpha, \beta)$.

- $F(x) = \int_{-\infty}^{x} f(x)dx = \frac{1}{\beta-\alpha} \int_{\alpha}^{x} 1dx = \frac{x-\alpha}{\beta-\alpha}$ for $\alpha \leq x \leq \beta$
- Differently from p.m.f.'s, densities can be larger than 1 (and arbitrarily large)
  - E.g., for $U(0, 0.5)$ we have $f(x) = 2$

**See R script**

# $X \sim Exp(\lambda)$

- For $X \sim Geo(p)$, we have: $F(x) = P(X \leq x) = 1 - (1-p)^{\lfloor x \rfloor}$ for $x \geq 0$
- extend to reals: $F(x) = P(X \leq x) = 1 - (1-p)^x = 1 - e^{x \cdot log(1-p)} = 1 - e^{-\lambda x}$
  for $\lambda = -log(1-p)$
- $f(x) = \frac{dF}{dx}(x) = \lambda e^{-\lambda x}$

DEFINITION. A continuous random variable has an *exponential distribution* with parameter $\lambda$ if its probability density function $f$ is given by $f(x) = 0$ if $x < 0$ and

$$f(x) = \lambda e^{-\lambda x} \quad \text{for } x \geq 0.$$

We denote this distribution by $Exp(\lambda)$.

- $\lambda$ is the rate of events in a Poisson point process, i.e., a process in which events occur continuously and independently at a constant average rate, e.g.,
  - $\lambda = {}^1/_{10}$ number of bus arrivals per minute, or ${}^1/_{\lambda} = 10$ minutes to wait for bus arrival
  - $P(X > 1) = 1 - P(X \leq 1) = e^{-\lambda} = 0.9048$ probability of waiting more than 1 minute.

# $X \sim Exp(\lambda)$

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- Plausible and empirically adequate model for:
  - time until a radioactive particle decays, time it takes before your next telephone call, ...
  - time until default (on payment to company debt holders) in reduced-form credit risk modeling, ...
  - time between animal roadkills, time between bank teller serves customers, ...
  - monthly and annual maximum values of daily rainfall, (some types of) surgery duration, ...
- Exponential is *memoryless*: $P(X > s + t | X > s) = e^{-\lambda \cdot (s+t)} / e^{-\lambda \cdot s} = e^{-\lambda \cdot t} = P(X > t)$

**See R script and seeing-theory.brown.edu**

# $X \sim N(\mu, \sigma^2)$

DEFINITION. A continuous random variable has a *normal distribution* with parameters $\mu$ and $\sigma^2 > 0$ if its probability density function $f$ is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad \text{for} \quad -\infty < x < \infty.$$

We denote this distribution by $N(\mu, \sigma^2)$.

- "Normal" means "typical" or "common"
- Also called Gaussian distribution, after **Carl Friedrich Gauss**
- Standard Normal/Gaussian is $N(0, 1)$
  - $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ sometimes written as $\phi(x)$
  - No closed form for $F(a) = \Phi(a) = \int_{-\infty}^{a} \phi(x)dx$
- Binomial approximation by a Normal distribution
  - $Bin(n, p) \approx N(np, np(1-p))$ for $n$ large and $0 \ll p \ll 1$ [**De Moivre–Laplace theorem**]

# CCDF of $Z \sim N(0,1)$

**Table B.1.** Right tail probabilities $1 - \Phi(a) = \mathrm{P}(Z \geq a)$ for an $N(0,1)$ distributed random variable $Z$.

| $a$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 5000 | 4960 | 4920 | 4880 | 4840 | 4801 | 4761 | 4721 | 4681 | 4641 |
| 0.1 | 4602 | 4562 | 4522 | 4483 | 4443 | 4404 | 4364 | 4325 | 4286 | 4247 |
| 0.2 | 4207 | 4168 | 4129 | 4090 | 4052 | 4013 | 3974 | 3936 | 3897 | 3859 |
| 0.3 | 3821 | 3783 | 3745 | 3707 | 3669 | 3632 | 3594 | 3557 | 3520 | 3483 |
| 0.4 | 3446 | 3409 | 3372 | 3336 | 3300 | 3264 | 3228 | 3192 | 3156 | 3121 |
| 0.5 | 3085 | 3050 | 3015 | 2981 | 2946 | 2912 | 2877 | 2843 | 2810 | 2776 |
| 0.6 | 2743 | 2709 | 2676 | 2643 | 2611 | 2578 | 2546 | 2514 | 2483 | 2451 |
| 0.7 | 2420 | 2389 | 2358 | 2327 | 2296 | 2266 | 2236 | 2206 | 2177 | 2148 |
| 0.8 | 2119 | 2090 | 2061 | 2033 | 2005 | 1977 | 1949 | 1922 | 1894 | 1867 |
| 0.9 | 1841 | 1814 | 1788 | 1762 | 1736 | 1711 | 1685 | 1660 | 1635 | 1611 |
| 1.0 | 1587 | 1562 | 1539 | 1515 | 1492 | 1469 | 1446 | 1423 | 1401 | 1379 |

- E.g., $P(Z > 1.04) = 0.1492$
- And in general for $X \sim N(\mu, \sigma^2)$?
  - Use identity $P(X \geq a) = P(Z \geq \frac{a-\mu}{\sigma})$ *[Proof in future lessons]*

**See R script**

# Quantiles

Definition. Let $X$ be a continuous random variable and let $p$ be a number between 0 and 1. The $p$th *quantile* or 100$p$th *percentile* of the distribution of $X$ is the smallest number $q_p$ such that

$$F(q_p) = \mathrm{P}(X \leq q_p) = p.$$

The *median* of a distribution is its 50th percentile.

- Median $m_X$ is $q_{0.5}$
- If $F()$ is *strictly* increasing, $q_p = F^{-1}(p)$
- E.g., for $Exp(\lambda)$, $F(a) = 1 - e^{-\lambda x}$, hence $F^{-1}(p) = \frac{1}{\lambda} \log \frac{1}{(1-p)}$

**See R script**

- General definition (also for discrete r.v.):

$$q_p = \inf_x \{P(X \leq x) \geq p\}$$

# Joint distributions: continuous random variables

DEFINITION. Random variables $X$ and $Y$ have a *joint continuous distribution* if for some function $f : \mathbb{R}^2 \to \mathbb{R}$ and for all numbers $a_1, a_2$ and $b_1, b_2$ with $a_1 \leq b_1$ and $a_2 \leq b_2$,

$$\mathrm{P}(a_1 \leq X \leq b_1, a_2 \leq Y \leq b_2) = \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(x, y)\, \mathrm{d}x\, \mathrm{d}y.$$

The function $f$ has to satisfy $f(x, y) \geq 0$ for all $x$ and $y$, and $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, \mathrm{d}x\, \mathrm{d}y = 1$. We call $f$ the *joint probability density function* of $X$ and $Y$.

- The marginal density functions of $X$ and $Y$ are:

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy \qquad f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx$$

- Moreover, as in the univariate case:

$$F(a, b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f(x, y) dxdy \qquad f(x, y) = \frac{d}{dx}\frac{d}{dy} F(x, y) = \frac{d^2}{dxdy} F(x, y)$$

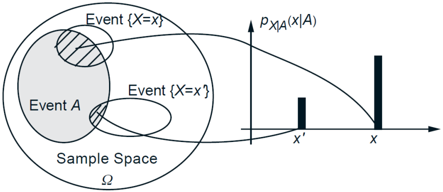
**See R script**

# Recalling conditional distribution

## Conditional distribution

Consider the joint distribution $P_{XY}$ of $X$ and $Y$. The conditional distribution of $X$ given $Y \in B$ with $P_Y(Y \in B) > 0$, is the function $F_{X|Y\in B} : \mathbb{R} \to [0,1]$:

$$F_{X|Y\in B}(a) = P_{X|Y}(X \leq a | Y \in B) = \frac{P_{XY}(X \leq a, Y \in B)}{P_Y(Y \in B)} \quad \text{for } -\infty < a < \infty$$




- Distribution of $X$ after knowing $Y \in B$.
- Chain rule: $P_{XY}(X \leq a, Y \in B) = P_{X|Y}(X \leq a|Y \in B)P_Y(Y \in B)$
- What if the distribution does not change w.r.t. the prior $P_X$?

# Independence of two random variables

## Independence $X \perp\!\!\!\perp Y$

A random variable $X$ is independent from a random variable $Y$, if for all $P(Y \leq b) > 0$:

$$P_{X|Y}(X \leq a | Y \leq b) = P_X(X \leq a) \quad \text{for } -\infty < a < \infty$$

- Properties
  - $X \perp\!\!\!\perp Y$ iff $P_{XY}(X \leq a, Y \leq b) = P_X(X \leq a) \cdot P_Y(Y \leq b) \quad \text{for } -\infty < a, b < \infty$
  - $X \perp\!\!\!\perp Y$ iff $Y \perp\!\!\!\perp X$ *[Symmetry]*
- For $X, Y$ **continuous** random variables:
  - $X \perp\!\!\!\perp Y$ iff $f_{XY}(x, y) = f_X(x) \cdot f_Y(y) \quad \text{for } -\infty < x, y < \infty$
  - **Exercise at home.** Prove it!
  - $X \perp\!\!\!\perp Y$ iff $P_{XY}(X \in \mathcal{A}, Y \in \mathcal{B}) = P_X(X \in \mathcal{A}) \cdot P_Y(Y \in \mathcal{B}) \quad \text{for } \mathcal{A}, \mathcal{B} \subseteq \mathbb{R}$ integrable

**See R script**

# Independence of multiple random variables

## Independence (factorization formula)

Random variables $X_1, \ldots, X_n$ are independent, if:

$$P(X_1 \leq a_1, \ldots, X_n \leq a_n) = \prod_{i=1}^{n} P(X_i \leq a_i) \quad \text{for } -\infty < a_1, \ldots, a_n < \infty$$

- $X_1, \ldots, X_n$ **continuous** random variables are independent iff:

$$f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i) \quad \text{for } -\infty < x_1, \ldots, x_n < \infty$$

- **Definition:** $X_1, \ldots, X_n$ are **i.i.d.** (independent and identically distributed) if $X_1, \ldots, X_n$ are independent and $X_i \sim F$ for $i = 1, \ldots, n$ for some distribution $F$

# Sum of independent continuous random variables

ADDING TWO INDEPENDENT CONTINUOUS RANDOM VARIABLES. Let $X$ and $Y$ be two independent continuous random variables, with probability density functions $f_X$ and $f_Y$. Then the probability density function $f_Z$ of $Z = X + Y$ is given by

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y) \, \mathrm{d}y$$

for $-\infty < z < \infty$.

- The integral is called the **convolution** of $f_X()$ and $f_Y()$
- $X, Y \sim Exp(\lambda)$, $Z = X + Y$, $\quad X, Y, Z \geq 0$ implies $0 \leq Y \leq Z$

$$f_Z(z) = \int_{-\infty}^{\infty} \lambda e^{-\lambda(z-y)} \lambda e^{-\lambda y} dy = \lambda^2 e^{-\lambda z} \int_0^z 1 dy = \lambda^2 e^{-\lambda z} z$$

- $Z = X_1 + \ldots + X_n$ for $X_i \sim Exp(\lambda)$ independent: *[Earlang $Erl(n, \lambda)$ distribution]*

$$f_Z(z) = \frac{\lambda(\lambda z)^{n-1} e^{-\lambda z}}{(n-1)!}$$

# $Gam(\alpha, \lambda)$

- Let $\lambda$ be some average rate of an event, e.g., $\lambda = 1/10$ number of buses in a minute
- The waiting times to see an event is Exponentially distributed. E.g., probability of waiting $x$ minutes to see one bus.
- The waiting times between $n$ occurrences of an event are Erlang distributed. E.g., probability of waiting $z$ minutes to see $n$ buses.

DEFINITION. A continuous random variable $X$ has a *gamma distribution* with parameters $\alpha > 0$ and $\lambda > 0$ if its probability density function $f$ is given by $f(x) = 0$ for $x < 0$ and

$$f(x) = \frac{\lambda (\lambda x)^{\alpha-1} \mathrm{e}^{-\lambda x}}{\Gamma(\alpha)} \qquad \text{for } x \geq 0,$$

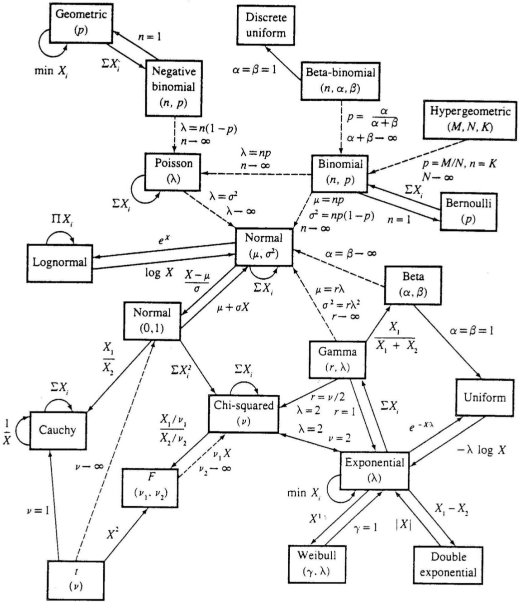
where the quantity $\Gamma(\alpha)$ is a normalizing constant such that $f$ integrates to 1. We denote this distribution by $Gam(\alpha, \lambda)$.

- Extends $Erl(n, \lambda)$ from $n \in \mathbb{N}$ to $\alpha \in \mathbb{R}^+$ by Euler's $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$ $\qquad [\Gamma(n) = (n-1)!]$
- Plausible and empirically adequate model for:
  - size of insurance claims, size of rainfalls, age distribution of cancer incidence, ...

**See R script**

# Common distributions

- **Probability distributions at Wikipedia**
- **Probability distributions in R**
- C. Forbes, M. Evans,
  N. Hastings, B. Peacock (2010)
  Statistical Distributions, 4th Edition
  Wiley




**Relationships among common distributions.** Solid lines represent transformations and special cases, dashed lines represent limits. Adapted from Leemis (1986).

# The continuous Bayes' rule

BAYES' RULE. Suppose the events $C_1, C_2, \ldots, C_m$ are disjoint and $C_1 \cup C_2 \cup \cdots \cup C_m = \Omega$. The conditional probability of $C_i$, given an arbitrary event $A$, can be expressed as:

$$P(C_i \mid A) = \frac{P(A \mid C_i) \cdot P(C_i)}{P(A \mid C_1)P(C_1) + P(A \mid C_2)P(C_2) + \cdots + P(A \mid C_m)P(C_m)}.$$

- **Definition.** Conditional density of $X$ given $Y = y$ with $f_Y(y) > 0$:

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

- Continuous Bayes' rule:

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)f_X(x)}{f_Y(y)} = \frac{f_{Y|X}(y|x)f_X(x)}{\int_{-\infty}^{\infty} f_{Y|X}(y|t)f_X(t)dt}$$

- **Exercise at home.** A light bulb has a life-time $X \sim Exp(\lambda)$. $\lambda$ is known to be $\sim U(1, 1.5)$. What can we say about the distribution of $\lambda$ give observed life-time $x$? Code your solution also in R.