Master Program in *Data Science and Business Informatics*

Statistics for Data Science

Lesson 14 - Law of large numbers, and the central limit theorem

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Markov's inequality

Notation. Indicator variable: $\mathbb{1}_{X \in A}(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$

- $E[\mathbb{1}_{X \in A}] = \sum_x \mathbb{1}_{X \in A}(x) p_X(a) = \sum_{x \in A} p_X(a) = P_X(X \in A) = P(\mathbb{1}_{X \in A} = 1)$

- Question: how much probability mass is near the expectation?

  Markov's inequality. For X non-negative (i.e., $P(X < 0) = 0$) and $\alpha > 0$:

$$P(X \geq \alpha) \leq \frac{E[X]}{\alpha}$$

  Proof. Take expectations of $\alpha \mathbb{1}_{X \geq \alpha} \leq X$. □

- For a non-negative r.v., the probability of a large value is inversely proportional to the value

  Corollary. For X non-negative, $E[X] > 0$ and $k > 0$: $P(X \geq kE[X]) \leq \frac{1}{k}$

Chebyshev's inequality

- Question: how much probability mass is near the expectation?

> CHEBYSHEV'S INEQUALITY. For an arbitrary random variable Y and any $a > 0$:
>
> $$P(|Y - E[Y]| \geq a) \leq \frac{1}{a^2} \text{Var}(Y) .$$

Proof. Let $X = (Y - E[Y])^2$ and $\alpha = a^2$. By Markov's inequality:

$$P(|Y - E[Y]| \geq a) = P((Y - E[Y])^2 \geq a^2) \leq \frac{E[(Y - E[Y])^2]}{a^2} = \frac{1}{a^2} Var(Y)$$

□

Chebyshev's inequality

- **"$\mu \pm$ a few σ" rule:** Most of the probability mass of a random variable is within a few standard deviations from its expectation!
- Let $\mu = E[Y]$ and $\sigma^2 = Var(Y) > 0$. For $k > 0$ (and hence $a = k\sigma > 0$):

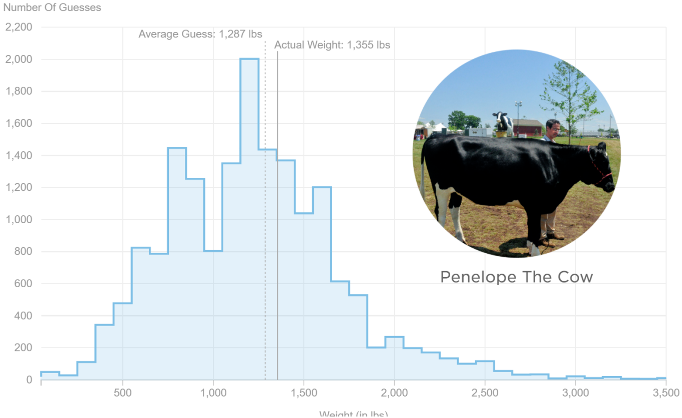
$$P(|Y - \mu| < k\sigma) = 1 - P(|Y - \mu| \geq k\sigma) \geq 1 - \frac{1}{k^2\sigma^2} Var(Y) = 1 - \frac{1}{k^2}$$

- For $k = 2, 3, 4$, the RHS is $3/4, 8/9, 15/16$
- Chebyshev's inequality is sharp when nothing is known about X, but in general it is a large bound!

See R script

Averages vary less

- Guessing the weight of a cow



- **See Francis Galton** (inventor of standard deviation and much more)

Expectation and variance of an average

- Let $X_1, X_2, \ldots, X_n$ be independent r. v. for which $E[X_i] = \mu$ and $Var(X_i) = \sigma^2$

$$\bar{X}_n = \frac{X_1 + X_2 + \ldots + X_n}{n}$$

Expectation and variance of an average. If $\bar{X}_n$ is the average of n independent random variables with the same expectation μ and variance σ^2, then

$$\mathrm{E}\left[\bar{X}_n\right] = \mu \quad \text{and} \quad \mathrm{Var}(\bar{X}_n) = \frac{\sigma^2}{n}.$$

- Notice that $X_1, \ldots, X_n$ are not required to be identically distributed!

See R script

The (weak) law of large numbers

- Apply Chebyshev's inequality to $\bar{X}_n$

$$P(|\bar{X}_n - \mu| > \epsilon) \leq \frac{1}{\epsilon^2} Var(\bar{X}_n) = \frac{\sigma^2}{n\epsilon^2}$$

- For $n \to \infty$, $\sigma^2/(n\epsilon^2) \to 0$

> THE LAW OF LARGE NUMBERS. If $\bar{X}_n$ is the average of n independent random variables with expectation μ and variance σ^2, then for any $\varepsilon > 0$:
>
> $$\lim_{n\to\infty} \mathrm{P}\left(|\bar{X}_n - \mu| > \varepsilon\right) = 0.$$

- probability that $\bar{X}_n$ is far from μ tends to 0 as $n \to \infty$! *[**Convergence in probability**]*
- It holds also if σ^2 is infinite (proof not included)
- Notice (again!) that $X_1, \ldots, X_n$ are not required to be identically distributed!

Recovering probability of an event

Objective: Let $C = (a, b]$, and want to know $p = P(X \in C)$

- Run n independent measurements
- Model the results as $X_1, \ldots, X_n$ random variables
- Define the indicator variables, for $i = 1, \ldots, n$:

$$Y_i = \mathbb{1}_{X_i \in C} = \begin{cases} 1 & \text{if } X_i \in C \\ 0 & \text{if } X_i \notin C \end{cases}$$

- Y_i's are independent *[Propagation of independence]*
- $E[Y_i] = 1 \cdot P(X_i \in C) + 0 \cdot P(X_i \notin C) = p$
- Defined $\bar{Y}_n = \frac{Y_1 + \ldots + Y_n}{n}$, by the law of large numbers:

$$\lim_{n \to \infty} P(|\bar{Y}_n - p| > \epsilon) = 0$$

- Frequency counting of $v \in (a, b]$ (e.g., in histograms) is a probability estimation method!

Estimating conditional probability

Objective: estimate $P(Y = y|A = a)$ given $(a_1, y_1), \ldots, (a_t, y_t)$ with $y \in \{0, 1, \ldots, k-1\}$

- Let $n = |\{(a_i, y_i) \mid a_i = a\}$ and $n_y = |\{(a_i, y_i) \mid a_i = a, y_i = y\}$
- Use $^{n_y}/_n$, the proportion of $Y = y$ over $A = a$: Ok for $n \to \infty$, not for n small
- m-estimate:

$$\frac{n_y + mp_y}{n + m}$$

  where m is a weight factor and $p_y = {}^{t_y}/_t$ prior probability with $t_y = |\{(a_i, y_i) \mid y_i = y\}$
- Smoothing regularization

$$\lambda(n)\frac{n_y}{n} + (1 - \lambda(n))p_y$$

  where $\lambda(n) \in [0, 1]$ is increasing with n
  - Interpolate $P(Y = y|A = a)$ with $P(Y = y)$
  - For $\lambda(n) = {}^{n}/_{(n+m)}$, we get the m-estimate
- Sample usage: target encoding of categorical attributes [Micci-Barreca, 2001]

See R script

Hoeffding bound

Theorem (Hoeffding bound)

If $\bar{X}_n$ is the average of n independent r.v. with expectation μ and $P(a \leq X_i \leq b) = 1$, then for any $\epsilon > 0$

$$P(|\bar{X}_n - \mu| \geq \epsilon) \leq 2e^{-2n\epsilon^2/(b-a)^2}$$

- For bounded support, a tight upper bound!
- When $a = 0, b = 1$ (e.g., Bernoulli trials):

$$P(|\bar{X}_n - \mu| \geq \epsilon) \leq 2e^{-2n\epsilon^2}$$

Corollary. If $\bar{X}_n$ is the average of n independent r.v. with expectation μ and $P(a \leq X_i \leq b) = 1$, then for any $n \geq {}^{1}/_{2\epsilon^2} \log {}^{2}/_{\delta}$: $P(|\bar{X}_n - \mu| \leq \epsilon) \geq 1 - \delta$

- ϵ accuracy: allowed error in estimation
- δ confidence: allowed probability of failure in achieving the accuracy

- E.g., recovering probability of an event: $P(|\bar{Y}_n - p| \leq 0.01) \geq 0.99$ for $n = 3516$

The central limit theorem

- Let $X_1, X_2, \ldots, X_n$ be independent r. v. for which $E[X_i] = \mu$ and $Var(X_i) = \sigma^2$

$$\bar{X}_n = \frac{X_1 + X_2 + \ldots + X_n}{n} \quad E[\bar{X}_n] = \mu \quad Var(\bar{X}_n) = \frac{\sigma^2}{n}$$

- Can we derive the distribution of $\bar{X}_n$?
- We already showed that, for $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ indepedent:

$$\frac{X_1 + X_2}{2} \sim N(\frac{\mu_1 + \mu_2}{2}, \frac{\sigma_1^2 + \sigma_2^2}{2^2})$$

- Assume $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$:

$$\bar{X}_n \sim N(\mu, \frac{\sigma^2}{n}) \qquad Z_n = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} = \frac{\bar{X}_n - E[\bar{X}_n]}{\sqrt{\frac{Var(\bar{X}_n)}{n}}} \sim N(0, 1)$$

- Interestingly, the same conclusion extends to any distribution for the X_i's!

The central limit theorem

THE CENTRAL LIMIT THEOREM. Let $X_1, X_2, \ldots$ be any sequence of independent identically distributed random variables with finite positive variance. Let μ be the expected value and σ^2 the variance of each of the X_i. For $n \geq 1$, let Z_n be defined by

$$Z_n = \sqrt{n}\,\frac{\bar{X}_n - \mu}{\sigma};$$

then for any number a

$$\lim_{n\to\infty} F_{Z_n}(a) = \Phi(a),$$

where Φ is the distribution function of the $N(0,1)$ distribution. In words: the distribution function of Z_n converges to the distribution function Φ of the standard normal distribution.

- It extends to not identically distributed r.v.'s [**Lindeberg's condition**]
- Why is it so frequent to observe a normal distribution?
  - Sometime it is the average/sum effects of other variables, e.g., as in "noise"
  - This justifies the common use of it to stand in for the effects of unobserved variables

See R script and **seeing-theory.brown.edu**

Applications: approximating probabilities

- Let $X_1, \ldots, X_n \sim Exp(2)$, for $n = 100$ $\mu = \sigma = {}^1\!/_2$
- Assume to observe realizations $x_1, \ldots, x_n$ such that $\bar{x}_n = \frac{1}{n}\sum_{i=1}^n x_i = 0.6$
- What is the probability $P(\bar{X}_n \geq 0.6)$ of observing such a value or a greater value?

Option A: Compute the distribution of $\bar{X}_n$

- $S_n = X_1 + \ldots + X_n \sim Erl(n, 2)$
- $\bar{X}_n = {}^{S_n}\!/_n$ hence by change-of-units transformation

$$F_{\bar{X}_n}(x) = F_{S_n}(n \cdot x) \quad \text{and} \quad f_{\bar{X}_n}(x) = n \cdot f_{S_n}(n \cdot x)$$

- and then:

$$P(\bar{X}_n \geq 0.6) = 1 - F_{\bar{X}_n}(0.6) = 1 - F_{S_n}(n \cdot 0.6) = 1 - \texttt{pgamma(60, n, 2)} = 0.0279$$

Applications: approximating probabilities

- Let $X_1, \ldots, X_n \sim Exp(2)$, for $n = 100$ $\mu = \sigma = {}^1/_2$
- Assume to observe realizations $x_1, \ldots, x_n$ such that $\bar{x}_n = \frac{1}{n}\sum_{i=1}^n x_i = 0.6$
- What is the probability $P(\bar{X}_n \geq 0.6)$ of observing such a value or a greater value?

Option B: Approximate them by using the CLT

- $Z_n = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$ implies $\bar{X}_n = \frac{\sigma}{\sqrt{n}} Z_n + \mu \sim N(\mu, {}^{\sigma^2}/_{n})$ for $n \to \infty$
- and then:

$$P(\bar{X}_n \geq 0.6) = P(\frac{\sigma}{\sqrt{n}} Z_n + \mu \geq 0.6) = P(Z_n \geq \frac{0.6 - \mu}{\sigma/\sqrt{n}}) \approx 1 - \Phi(\frac{0.6 - 0.5}{0.5/10}) == 0.0228$$

- also, notice $X_1 + \ldots + X_n = \sqrt{n}\sigma Z_n + n\mu \sim N(n\mu, n\sigma^2)$

See R script

How large should n be?

- How fast is the convergence of Z_n to $N(0, 1)$?
- The approximation might be poor when:
  - n is small
  - X_i is asymmetric, bimodal, or discrete
  - the value to test (0.6 in our example) is far from μ

the myth of $n \geq 30$

Optional reference

Daniele Micci-Barreca (2001)
A Preprocessing Scheme for High-Cardinality Categorical Attributes in Classification and Prediction Problems
SIGKDD Explor. Newsl. 3 (1), 27 – 32.