Master Program in *Data Science and Business Informatics*

Statistics for Data Science

Lesson 18 - Unbiased estimators. Efficiency and MSE

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Statistical model for repeated measurement

- A dataset $x_1, \ldots, x_n$ consists of repeated measurements of a phenomenon we are interested in understanding
  - E.g., measurement of the speed of light
- We model a dataset as the realization of a random sample

Random sample

A *random sample* is a collection of i.i.d. random variables $X_1, \ldots, X_n \sim F(\alpha)$, where $F()$ is the distribution and α its parameter(s).

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- Challenging questions/inferences on a population given a sample:
  - How to determine $E[X]$, $Var(X)$, or other functions of X?
  - How to determine α, assuming to know the form of F?
  - How to determine both F and α?

An example

Table 17.1. Michelson data on the speed of light.

850	740	900	1070	930	850	950	980	980	880
1000	980	930	650	760	810	1000	1000	960	960
960	940	960	940	880	800	850	880	900	840
830	790	810	880	880	830	800	790	760	800
880	880	880	860	720	720	620	860	970	950
880	910	850	870	840	840	850	840	840	840
890	810	810	820	800	770	760	740	750	760
910	920	890	860	880	720	840	850	850	780
890	840	780	810	760	810	790	810	820	850
870	870	810	740	810	940	950	800	810	870

- What is an estimate of the true speed of light (estimand)?

$$x_1 = 850, \text{ or } \mathit{min}\ x_i, \text{ or } \mathit{max}\ x_i, \text{ or } \bar{x}_n = 852.4\ ?$$

An example

- Speed of light dataset as realization of

$$X_i = c + \epsilon_i$$

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- Use some info. For $X = X_1$:

$$E[X] = E[X_1] = c \qquad Var(X) = Var(X_1) = \sigma^2$$

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- How to estimate?
- Use some info. For $X = X_1$:

$$E[X] = E[X_1] = c \qquad Var(X) = Var(X_1) = \sigma^2$$

- Use all info. For $\bar{X}_n = (X_1 + \ldots + X_n)/n$:

$$E[\bar{X}_n] = c \qquad Var(\bar{X}_n) = \frac{Var(X_1)}{n} = \frac{\sigma^2}{n}$$

  Hence, for $n \to \infty$, $Var(\bar{X}_n) \to 0$

Estimate

Estimand and estimate

An estimand θ is an unknown parameter of a distribution $F()$.
An *estimate* t of θ is a value that obtained as a function $h()$ over a dataset $x_1, \ldots, x_n$:

$$t = h(x_1, \ldots, x_n)$$

- $t = \bar{x}_n = 852.4$ is an estimate of the speed of light (estimand) $t = x_1 = 850$ is another estimate

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- Since $x_1, \ldots, x_n$ are modelled as realizations of $X_1, \ldots, X_n$, estimates are realizations of the corresponding sample statistics $h(X_1, \ldots, X_n)$

Statistics and estimator

A statistics is a function of $h(X_1, \ldots, X_n)$ of r.v.'s.
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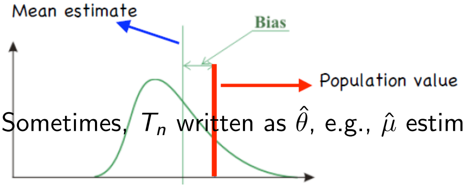
- An estimate $t = h(x_1, \ldots, x_n)$ is a realization of the estimator $T_n = h(X_1, \ldots, X_n)$
- $T_n = \bar{X}_n = (X_1 + \ldots, X_n)/n$ is an estimator of μ $T_n = X_1$ is another estimator

Unbiased estimator

- The probability distribution of an estimator T is called the *sampling distribution* of T



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An estimator $T_n = h(X_1, \ldots, X_n)$ of a parameter θ (estimand) is *unbiased* if:

$$E[T_n] = \theta$$

If the difference $E[T_n] - \theta$, called the *bias* of T_n, is non-zero, T_n is called a *biased* estimator.

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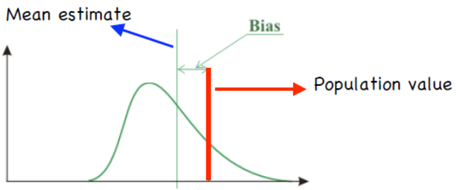

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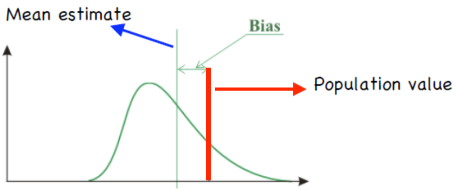

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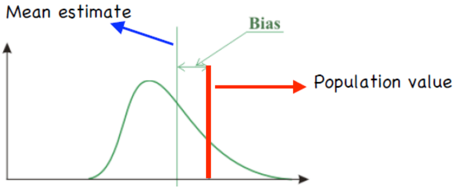
- $E[T_n] > \theta$ is a positive bias, $E[T_n] < \theta$ is a negative bias
- **Asymptotically unbiased:** $\lim_{n\to\infty} E[T_n] = \theta$
- Sometimes, T_n written as $\hat{\theta}$, e.g., $\hat{\mu}$ estimator of μ



On $E[T]$

- Random sample i.i.d. $X_1, \ldots, X_n \sim F(\alpha)$
- $E[T] = E[h(X_1, \ldots, X_n)]$ over the joint distribution $\prod_{i=1}^{n} F(\alpha)$
- E.g., for $F()$ continuous with d.f. $f()$

$$E[T] = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} h(x_1, \ldots, x_n) f(x_1) \ldots f(x_n) dx_1, \ldots, dx_n$$

When is an estimator better than another one?

- The standard deviation of the sampling distribution is called the *standard error* (SE)

Efficiency of unbiased estimators

Let T_1 and T_2 be unbiased estimators of the same parameter θ. The estimator T_2 is *more efficient* than T_1 if:

$$Var(T_2) < Var(T_1)$$

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- Speed of light example:
  - $E[X_1] = E[X_2] = \ldots = E[\bar{X}_n] = c$, i.e., all unbiased estimators

  The mean is more efficient than a single value

$$Var(\bar{X}_n) = \sigma^2/n < \sigma^2 = Var(X_1) \qquad \frac{Var(X_1)}{Var(\bar{X}_n)} = n$$

Unbiased estimators for expectation and variance

UNBIASED ESTIMATORS FOR EXPECTATION AND VARIANCE. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a distribution with finite expectation μ and finite variance σ^2. Then

$$\bar{X}_n = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

is an *unbiased estimator for* μ and

$$S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2$$

is an *unbiased estimator for* σ^2.

- Estimates: sample mean $\bar{x}_n$ and sample variance s_n^2
- $E[\bar{X}_n] = (E[X_1] + \ldots + E[X_n])/n = \mu$ and, by CLT, $Var(\bar{X}_n) \to 0$ for $n \to \infty$

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- $E[\bar{X}_n] = (E[X_1] + \ldots + E[X_n])/n = \mu$ and, by CLT, $Var(\bar{X}_n) \to 0$ for $n \to \infty$
- Why division by $n-1$ in S_n^2? *[Bessel's correction]*

$E[S_n^2] = \sigma^2$

(1) $E[X_i - \bar{X}_n] = E[X_i] - E[\bar{X}_n] = \mu - \mu = 0$

(2) $Var(X_i - \bar{X}_n) = E[(X_i - \bar{X}_n)^2] - E[X_i - \bar{X}_n]^2 = E[(X_i - \bar{X}_n)^2]$ *[by (1)]*

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- In general, $Var(S_n^2) = \frac{1}{n}(\mu_4 - \frac{n-3}{n-1}\sigma^4) \to 0$ for $n \to \infty$

Degree of freedom

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$$E[V_n^2] = E[\frac{n-1}{n}S_n^2] = \frac{n-1}{n}\sigma^2$$

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- The *degrees of freedom* for an estimate is the number of values minus the number of parameters already estimated
- Assume that μ is known. Show that $\frac{1}{n}\sum_{i=1}^n (X_i - \mu)^2$ is unbiased **[Prove it]**

Unbiasedness does not carry over (no functional invariance)

- $E[S_n^2] = \sigma^2$ implies $E[S_n] = \sigma$?
- Since $g(x) = x^2$ is convex, by Jensen's inequality:

$$\sigma^2 = E[S_n^2] = E[g(S_n)] > g(E[S_n]) = E[S_n]^2$$

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- In general, if T unbiased for θ does not imply $g(T)$ unbiased for $g(\theta)$
  - But it holds for $g()$ linear transformation
- A non-parametric (i.e., distribution free) unbiased estimator of σ **does not exist**

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$$\text{for } n \to \infty, T \sim N(m, \frac{1}{4nf(m)^2})$$

and then for $n \to \infty$:

$$E[Med(X_1, \ldots, X_n)] = m$$

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- $T = Quantile_p(X_1, \ldots, X_n)$, for X_i with density function $f(x)$

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$$\text{for } n \to \infty, T \sim N(m, \frac{1}{4nf(m)^2})$$

and then for $n \to \infty$:

$$E[Med(X_1, \dots, X_n)] = m$$

- $T = Quantile_p(X_1, \dots, X_n)$, for X_i with density function $f(x)$
- Let p quantile be the true quantile, i.e., $F(q) = p$: **[CLT for quantiles]**

$$\text{for } n \to \infty, T \sim N(q, \frac{p(1-p)}{nf(q)^2})$$

and then for $n \to \infty$:

$$E[Quantile_p(X_1, \dots, X_n)] = q$$

See R script

Estimator for MAD

- Median of absolute deviations (MAD):

$$T = MAD(X_1, \ldots, X_n) = Med(|X_1 - Med(X_1, \ldots, X_n)|, \ldots, |X_n - Med(X_1, \ldots, X_n)|)$$

  - For $X \sim F$, the population MAD is $Md = G^{-1}(0.5)$ where $|X - F^{-1}(0.5)| \sim G$
  - For F symmetric, $Md = F^{-1}(0.75) - F^{-1}(0.5)$.
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- Under mild assumptions: ***[CLT for MADs]***

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- Then, for $n \to \infty$:

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Estimators for correlation

- Pearson's r estimator:

$$r = \frac{\sum_{i=1}^{n}(X_i - \bar{X}) \cdot (Y_i - \bar{Y})}{\sqrt{\sum_{i=1}^{n}(X_i - \bar{X})^2 \cdot \sum_{i=1}^{n}(Y_i - \bar{Y})^2}} \qquad \rho = \frac{Cov(X, Y)}{\sigma_X \cdot \sigma_Y}$$

  - **Fisher transformation** $F(r) = arctanh(r) = \frac{1}{2} \log \frac{1+r}{1-r}$
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  Hence:

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- Same for Spearman's correlation (as it is a special case of Pearson's)

Estimators for correlation

- Kendall's τ_a estimator:

$$\tau_{xy} = \frac{2\sum_{i<j} sgn(X_i - X_j) \cdot sgn(Y_i - Y_j)}{n \cdot (n-1)} \qquad \theta = E[sgn(X_1 - X_2) \cdot sgn(Y_1 - Y_2)]$$

  - For $n > 10$, the sampling distribution is well approximated as:

$$\tau_{xy} \sim N(\theta, \frac{2(2n+5)}{9n(n-1)})$$

Hence:

$$E[\tau_{xy}] = \theta$$

See R script

Example: estimating the probability of zero arrivals

- $X_1, \ldots, X_n$, for $n = 30$, observations:

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- We want to estimate $p_0 = p(0)$, probability of zero arrivals
- Frequentist-based estimator S:

$$S = \frac{|\{i \mid X_i = 0\}|}{n}$$

  - Takes values $0/30, 1/30, \ldots, 30/30$ ... may not exactly be p_0
  - $S = Y/n$ where $Y = I_{X_1=0} + \ldots + I_{X_n=0} \sim Bin(n, p_0)$
  - Hence, $E[S] = \frac{1}{n}E[Y] = \frac{n}{n}p_0 = p_0$ *[S is unbiased]*

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    Hence T is biased!
  - $T = e^{-Z/n}$ where $Z = X_1 + \ldots + X_n$ is the sum of $Poi(\mu)$'s, hence $Z \sim Poi(n \cdot \mu)$

$$E[T] = \sum_{k=0}^{\infty} e^{-\frac{k}{n}} \frac{(n\mu)^k}{k!} e^{-n\mu} = e^{-n\mu(1-e^{-1/n})} \rightarrow e^{-\mu} = p_0 \text{ for } n \rightarrow \infty$$

    Hence T is asymptotically unbiased! *[Exercise 19.9]*

See R script

Example: estimating the probability of zero arrivals

- Let's look at the variances:

$$Var(S) = \frac{1}{n^2} Var(Y) = \frac{np_0(1 - p_0)}{n^2} = \frac{p_0(1 - p_0)}{n} \to 0 \text{ for } n \to \infty$$

$$Var(T) = E[T^2] - E[T]^2 = \dots \text{ exercise } \dots \to 0 \text{ for } n \to \infty$$

See R script

MSE: Mean Squared Error of an estimator

- What if one estimator is unbiased and the other is biased but with a smaller variance?

MSE

The Mean Squared Error of an estimator T for a parameter θ is defined as:

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- Hence, $MSE = Var + Bias^2$

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- Hence, $MSE = Var + Bias^2$
- A biased estimator with a small variance may be better than an unbiased one with a large variance!
- Squared error consistent estimator: $\lim_{n\to\infty} MSE(T_n) = 0$

See R script