Web Mining ed Analisi delle Reti Sociali

Proprietà delle Reti – Richiami di elementi di statistica

Dino Pedreschi
Dipartimento di Informatica
Università di Pisa
www.di.unipi.it/~pedre

"Natural" Networks and Universality

- Consider many kinds of networks:
  - social, technological, business, economic, content,…
- These networks tend to share certain *informal* properties:
  - large scale; continual growth
  - distributed, organic growth: vertices "decide" who to link to
  - interaction restricted to links
  - mixture of local and long-distance connections
  - abstract notions of distance: geographical, content, social,…
- Do natural networks share more *quantitative* universals?
- What would these "universals" be?
- How can we make them precise and measure them?
- How can we explain their universality?
- This is the domain of *social network theory*
- Sometimes also referred to as *link analysis*

Some Interesting Quantities

- *Connected components*:
  - how many, and how large?
- *Network diameter*:
  - maximum (worst-case) or average?
  - exclude infinite distances? (disconnected components)
  - the small-world phenomenon
- *Clustering*:
  - to what extent that links tend to cluster “locally”?
  - what is the balance between local and long-distance connections?
  - what roles do the two types of links play?
- *Degree distribution*:
  - what is the typical degree in the network?
  - what is the overall distribution?

The small-world effect

Consider an undirected network, and let us define ℓ to be the mean geodesic (i.e., shortest) distance between vertex pairs in a network:

$$\ell = \frac{1}{\frac{1}{2}n(n+1)} \sum_{i \geq j} d_{ij}, \tag{1}$$

where d_{ij} is the geodesic distance from vertex i to vertex j.

$$\ell^{-1} = \frac{1}{\frac{1}{2}n(n+1)} \sum_{i \geq j} d_{ij}^{-1}.$$

Transitivity – the clustering coefficient

In the language of social networks, the friend of your friend is likely also to be your friend. In terms of network topology, transitivity means the presence of a heightened number of triangles in the network—sets of three vertices each of which is connected to each of the others. It can be quantified by defining a clustering coefficient C thus:

$$C = \frac{3\times \text{ number of triangles in the network}}{\text{number of connected triples of vertices}}, \qquad (3)$$

where a "connected triple" means a single vertex with edges running to an unordered pair of others (see Fig. 5).

Transitivity – the clustering coefficient

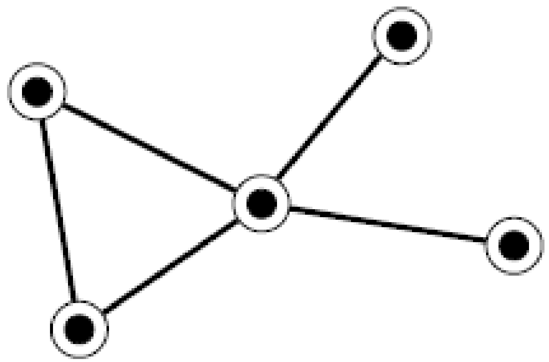


FIG. 5 Illustration of the definition of the clustering coefficient C, Eq. (3). This network has one triangle and eight connected triples, and therefore has a clustering coefficient of $3 \times 1/8 = \frac{3}{8}$. The individual vertices have local clustering coefficients, Eq. (5), of 1, 1, $\frac{1}{6}$, 0 and 0, for a mean value, Eq. (6), of $C = \frac{13}{30}$.

$$C = \frac{6\times \text{ number of triangles in the network}}{\text{number of paths of length two}}, \quad (4)$$

Transitivity – the clustering coefficient

An alternative definition of the clustering coefficient, also widely used, has been given by Watts and Strogatz [416], who proposed defining a local value

$$C_i = \frac{\text{number of triangles connected to vertex } i}{\text{number of triples centered on vertex } i}. \qquad (5)$$

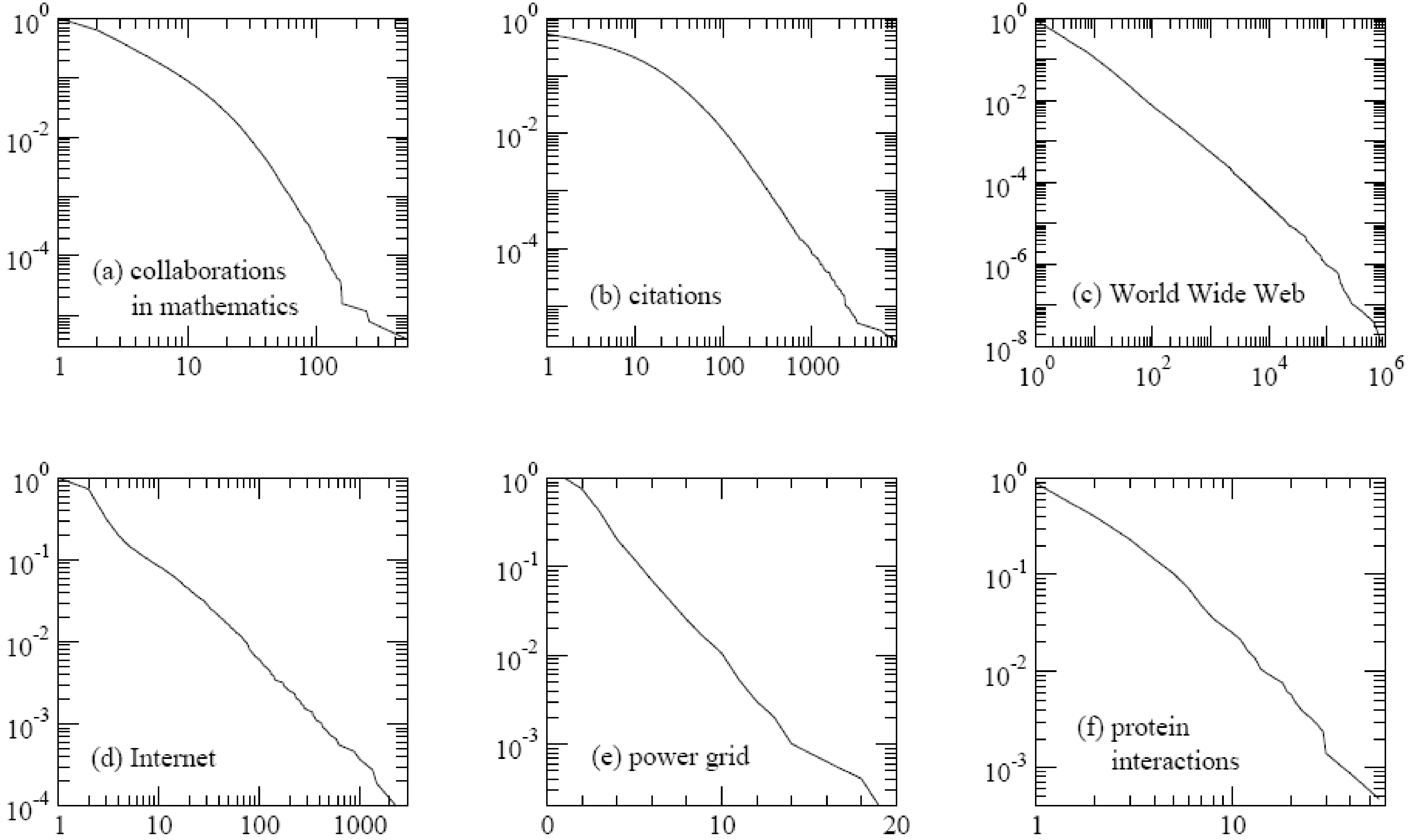
For vertices with degree 0 or 1, for which both numerator and denominator are zero, we put $C_i = 0$. Then the clustering coefficient for the whole network is the average

$$C = \frac{1}{n} \sum_i C_i. \qquad (6)$$

Degree distribution

- The **degree** of a vertex in a network is the number of edges incident on (i.e., connected to) that vertex.
- $\mathbf{p_k}$ = the fraction of vertices in the network that have degree k.
- Equivalently, $\mathbf{p_k}$ = the **probability** that a vertex chosen uniformly at random has **degree k**.
- A plot of $\mathbf{p_k}$ for any given network can be formed by a **histogram** of the degrees of vertices.
- This histogram is the **degree distribution** for the network

Degree distributions for six networks



Actor Connectivity (power law)

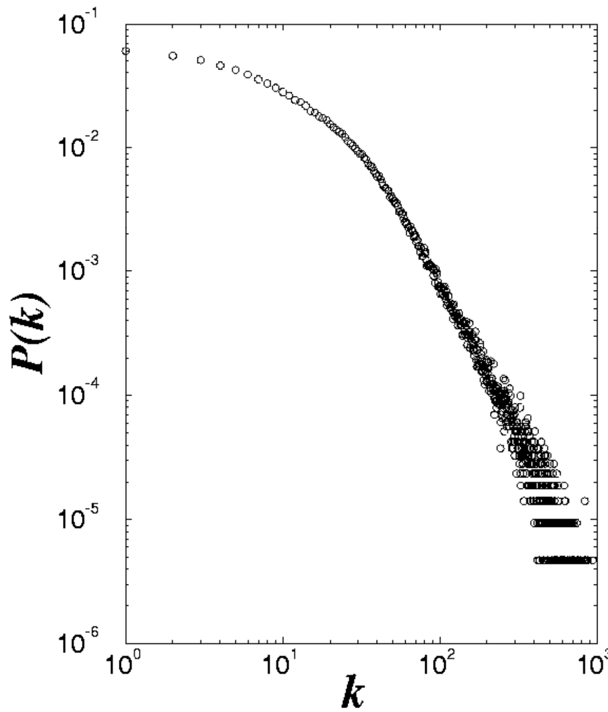
Days of Thunder (1990)
Far and Away (1992)
Eyes Wide Shut (1999)

N = 212,250 actors
$\langle k \rangle$ = 28.78

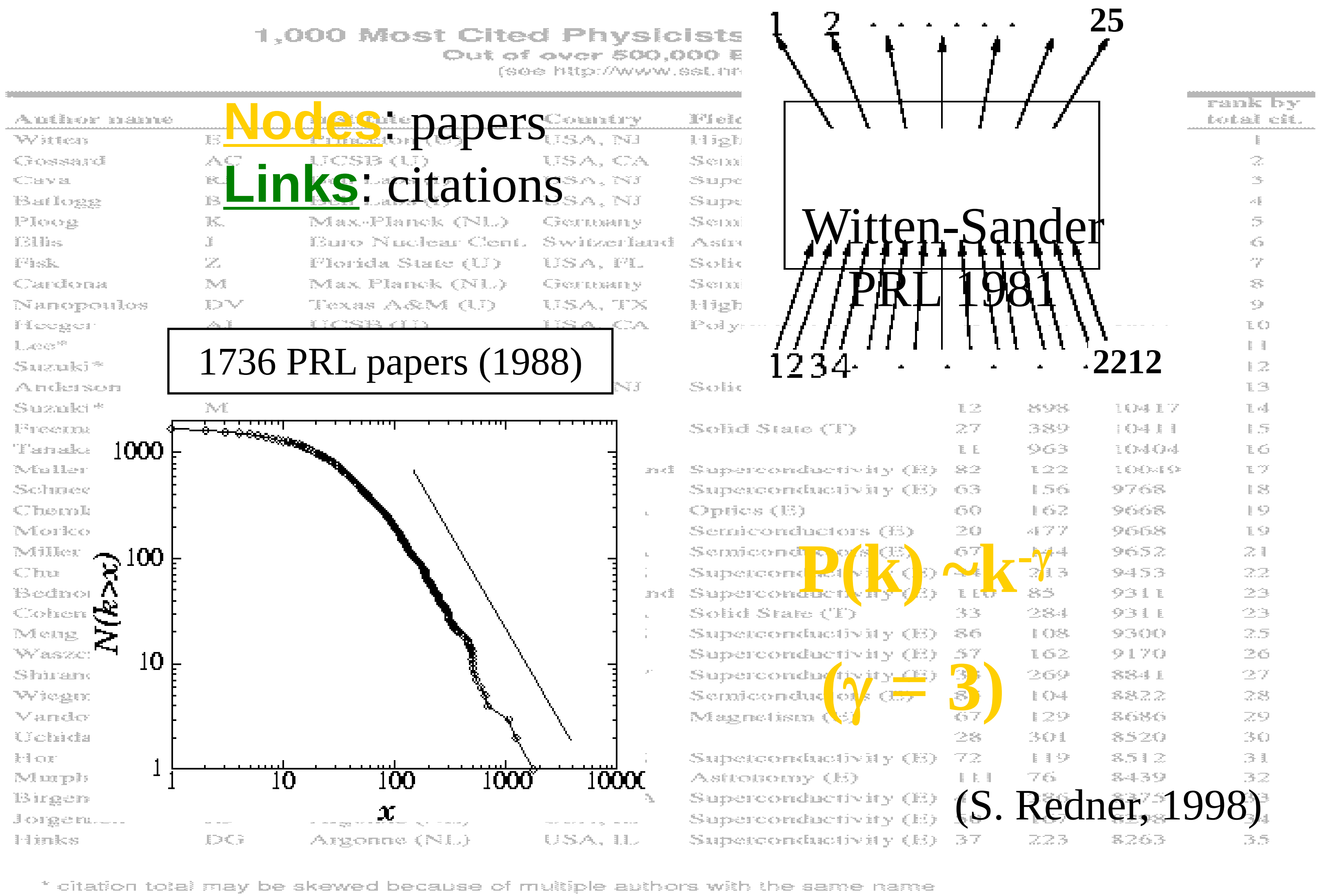
$P(k) \sim k^{-\gamma}$

γ=2.3

Nodes: actors
Links: cast jointly



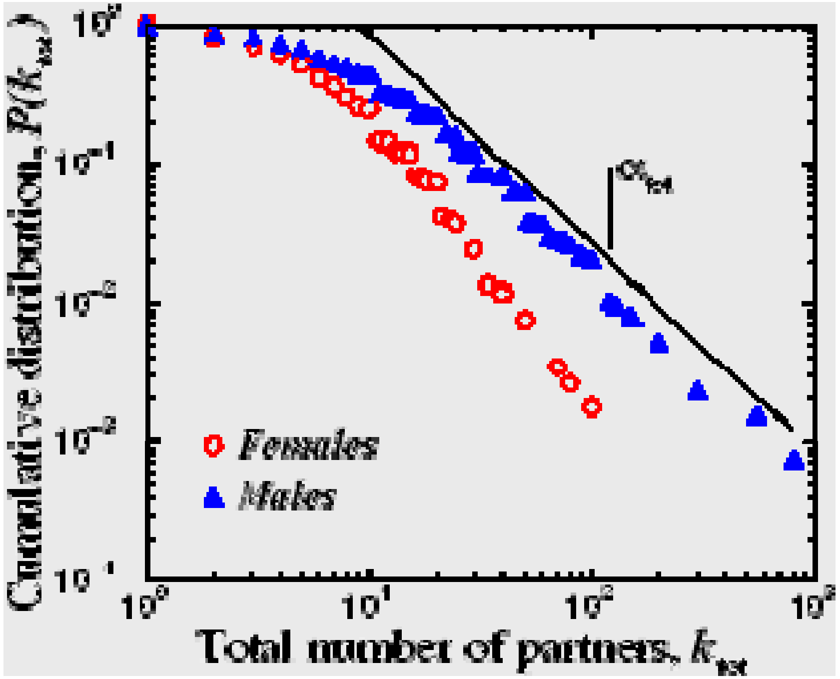
Science Citation Index (power law)



Sex-Web (power law)

Nodes: people (Females; Males)

Links: sexual relationships



4781 Swedes; 18-74;
59% response rate.
Liljeros et al. Nature 2001

Basic statisics for some published networks

	network	type	n	m	z	ℓ	α	$C^{(1)}$	$C^{(2)}$	r
social	film actors	undirected	449 913	25 516 482	113.43	3.48	2.3	0.20	0.78	0.208
	company directors	undirected	7 673	55 392	14.44	4.60	–	0.59	0.88	0.276
	math coauthorship	undirected	253 339	496 489	3.92	7.57	–	0.15	0.34	0.120
	physics coauthorship	undirected	52 909	245 300	9.27	6.19	–	0.45	0.56	0.363
	biology coauthorship	undirected	1 520 251	11 803 064	15.53	4.92	–	0.088	0.60	0.127
	telephone call graph	undirected	47 000 000	80 000 000	3.16		2.1			
	email messages	directed	59 912	86 300	1.44	4.95	1.5/2.0		0.16	
	email address books	directed	16 881	57 029	3.38	5.22	–	0.17	0.13	0.092
	student relationships	undirected	573	477	1.66	16.01	–	0.005	0.001	−0.029
	sexual contacts	undirected	2 810				3.2			
information	WWW `nd.edu`	directed	269 504	1 497 135	5.55	11.27	2.1/2.4	0.11	0.29	−0.067
	WWW Altavista	directed	203 549 046	2 130 000 000	10.46	16.18	2.1/2.7			
	citation network	directed	783 339	6 716 198	8.57		3.0/–			
	Roget's Thesaurus	directed	1 022	5 103	4.99	4.87	–	0.13	0.15	0.157
	word co-occurrence	undirected	460 902	17 000 000	70.13		2.7		0.44	
technological	Internet	undirected	10 697	31 992	5.98	3.31	2.5	0.035	0.39	0.189
	power grid	undirected	4 941	6 594	2.67	18.99	–	0.10	0.080	−0.003
	train routes	undirected	587	19 603	66.79	2.16	–		0.69	−0.033
	software packages	directed	1 439	1 723	1.20	2.42	1.6/1.4	0.070	0.082	−0.016
	software classes	directed	1 377	2 213	1.61	1.51	–	0.033	0.012	−0.119
	electronic circuits	undirected	24 097	53 248	4.34	11.05	3.0	0.010	0.030	−0.154
	peer-to-peer network	undirected	880	1 296	1.47	4.28	2.1	0.012	0.011	−0.366
biological	metabolic network	undirected	765	3 686	9.64	2.56	2.2	0.090	0.67	−0.240
	protein interactions	undirected	2 115	2 240	2.12	6.80	2.4	0.072	0.071	−0.156
	marine food web	directed	135	598	4.43	2.05	–	0.16	0.23	−0.263
	freshwater food web	directed	92	997	10.84	1.90	–	0.20	0.087	−0.326
	neural network	directed	307	2 359	7.68	3.97	–	0.18	0.28	−0.226

A “Canonical” Natural Network has…

- *Few* connected components:
  - often only 1 or a small number, indep. of network size
- *Small* diameter:
  - often a constant independent of network size (like 6)
  - or perhaps growing only logarithmically with network size or even shrink?
  - typically exclude infinite distances
- A *high* degree of clustering:
  - considerably more so than for a random network
  - in tension with small diameter
- A *heavy-tailed* degree distribution:
  - a small but reliable number of high-degree vertices
  - often of *power law* form

Probabilistic Models of Networks

- All of the network generation models we will study are *probabilistic* or *statistical* in nature
- They can generate networks of any size
- They often have various *parameters* that can be set:
  - size of network generated
  - average degree of a vertex
  - fraction of long-distance connections
- The models generate a *distribution* over networks
- Statements are always *statistical* in nature:
  - *with high probability*, diameter is small
  - on average, degree distribution has heavy tail
- Thus, we're going to need some basic statistics and probability theory

Social Network Analysis

- Social Network Introduction
- Statistics and Probability Theory
- Models of Social Network Generation
- Networks in Biological System
- Mining on Social Network
- Summary

Probability and Random Variables

- A *random variable* X is simply a variable that *probabilistically* assumes values in some set
  - set of possible values sometimes called the *sample space* S of X
  - sample space may be small and simple or large and complex
    - S = {Heads, Tails}, X is outcome of a coin flip
    - S = {0,1,…,U.S. population size}, X is number voting democratic
    - S = all networks of size N, X is generated by *preferential attachment*

- Behavior of X determined by its *distribution* (or *density*)
  - for each value x in S, specify Pr[X = x]
  - these probabilities sum to exactly 1 (mutually exclusive outcomes)
  - complex sample spaces (such as large networks):
    - distribution often defined *implicitly* by simpler components
    - might specify the probability that each *edge* appears independently
    - this *induces* a probability distribution over *networks*
    - may be difficult to *compute* induced distribution

Some Basic Notions and Laws

- *Independence*:
  - let X and Y be random variables
  - independence: for any x and y, Pr[X = x & Y = y] = Pr[X=x]Pr[Y=y]
  - intuition: value of X does not influence value of Y, vice-versa
  - dependence:
    - e.g. X, Y coin flips, but Y is always opposite of X
- *Expected (mean) value* of X:
  - only makes sense for *numeric* random variables
  - "average" value of X according to its distribution
  - formally, E[X] = Σ (Pr[X = x] X), sum is over all x in S
  - often denoted by μ
  - *always* true: E[X + Y] = E[X] + E[Y]
  - true only for *independent* random variables: E[XY] = E[X]E[Y]
- *Variance* of X:
  - Var(X) = E[(X – μ)^2]; often denoted by σ^2
  - *standard deviation* is sqrt(Var(X)) = σ
- *Union bound:*
  - for any X, Y, Pr[X=x & Y=y] <= Pr[X=x] + Pr[Y=y]

Convergence to Expectations

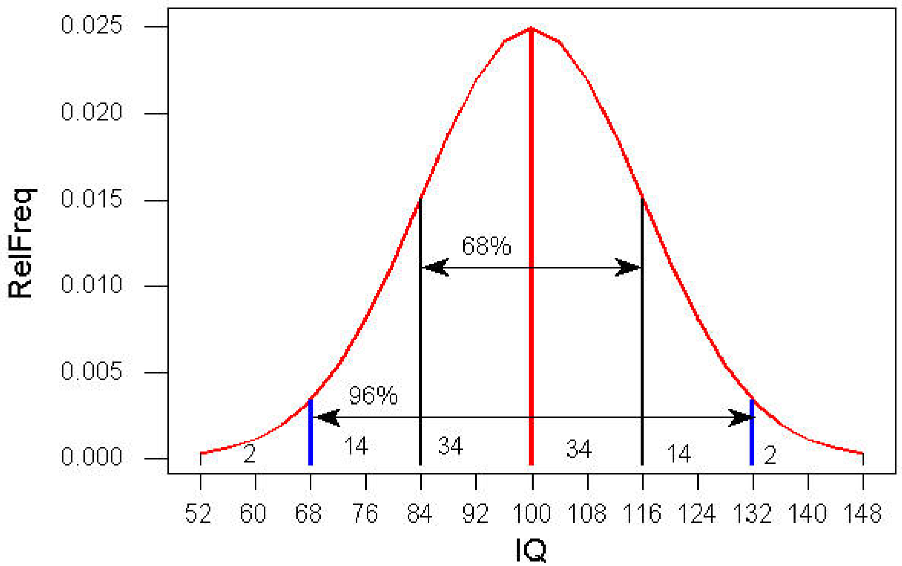
- Let $X_1, X_2, \ldots, X_n$ be:
  - *independent* random variables
  - with the *same* distribution Pr[X=x]
  - expectation μ = E[X] and variance σ^2
  - independent and identically distributed (i.i.d.)
  - essentially n repeated "trials" of the same experiment
  - natural to examine r.v. $Z = (1/n) \Sigma X_i$, where sum is over i=1,…,n
  - example: number of heads in a sequence of coin flips
  - example: degree of a vertex in the random graph model
  - E[Z] = E[X]; what can we say about the *distribution* of Z?

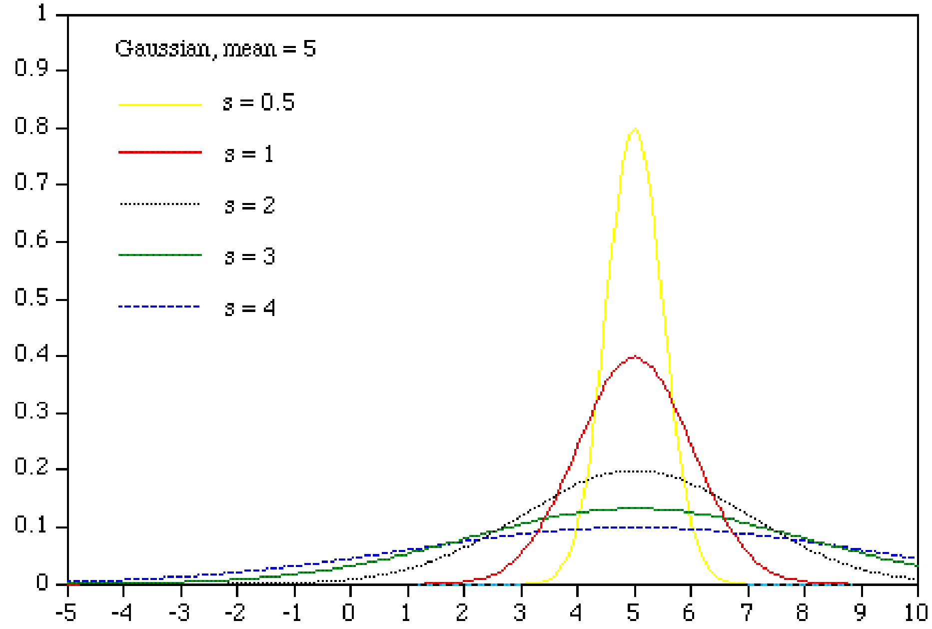
- *Central Limit Theorem:*
  - as n becomes large, Z becomes *normally distributed*
    - with expectation μ and variance σ^2/n

The Normal Distribution

- The *normal* or *Gaussian* density:
  - applies to continuous, real-valued random variables
  - characterized by mean (average) m and standard deviation s
  - *density* at x is defined as
    - $(1/(\sigma \, sqrt(2\pi))) \exp(-(x-\mu)^2/2\sigma^2)$
    - special case $\mu = 0$, $\sigma = 1$: a $\exp(-x^2/b)$ for some constants a,b > 0
  - peaks at $x = \mu$, then dies off *exponentially* rapidly
  - the classic "bell-shaped curve"
    - exam scores, human body temperature,
  - remarks:
    - can control mean and standard deviation independently
    - can make as "broad" as we like, but always have *finite variance*

The Normal Distribution



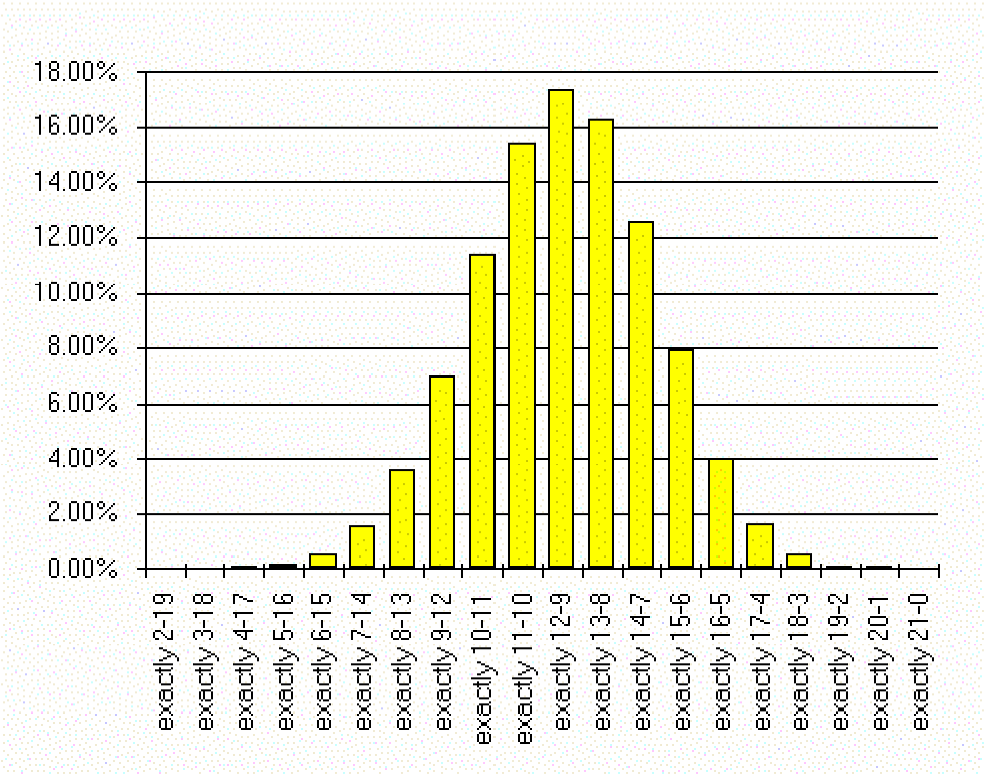


The Binomial Distribution

- coin with Pr[heads] = p, flip n times
- probability of getting exactly k heads:
  - choose(n,k) $p^k(1-p)^{n-k}$
- for large n and p *fixed:*
  - approximated well by a normal with

    $\mu = np$, $\sigma = \text{sqrt}(np(1-p))$
  - $\sigma/\mu \rightarrow 0$ as n grows
  - leads to strong large deviation bounds

The Binomial Distribution

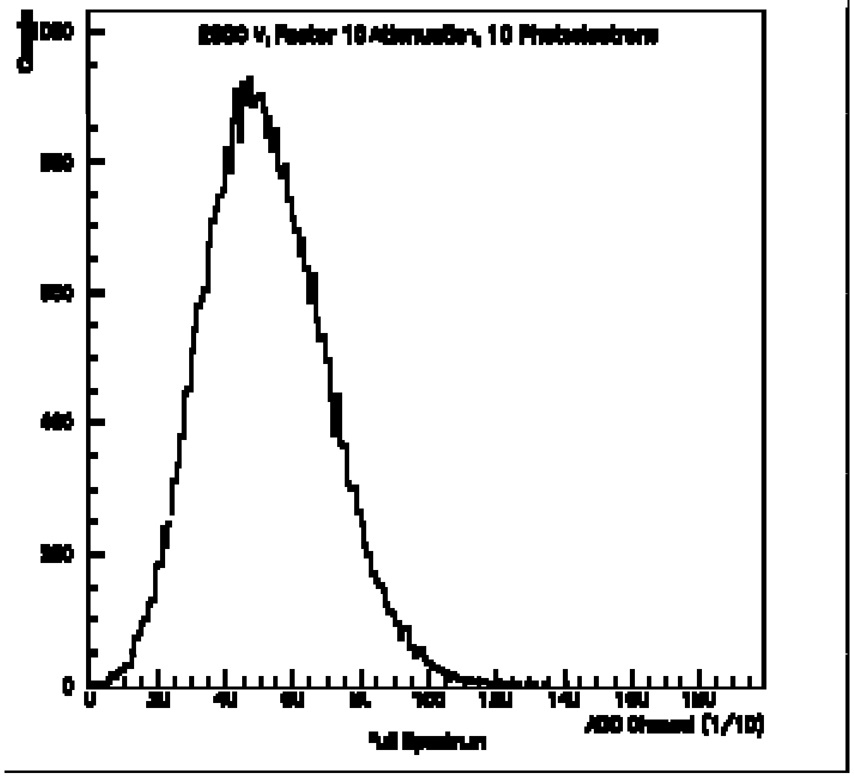


www.professionalgambler.com/ binomial.html

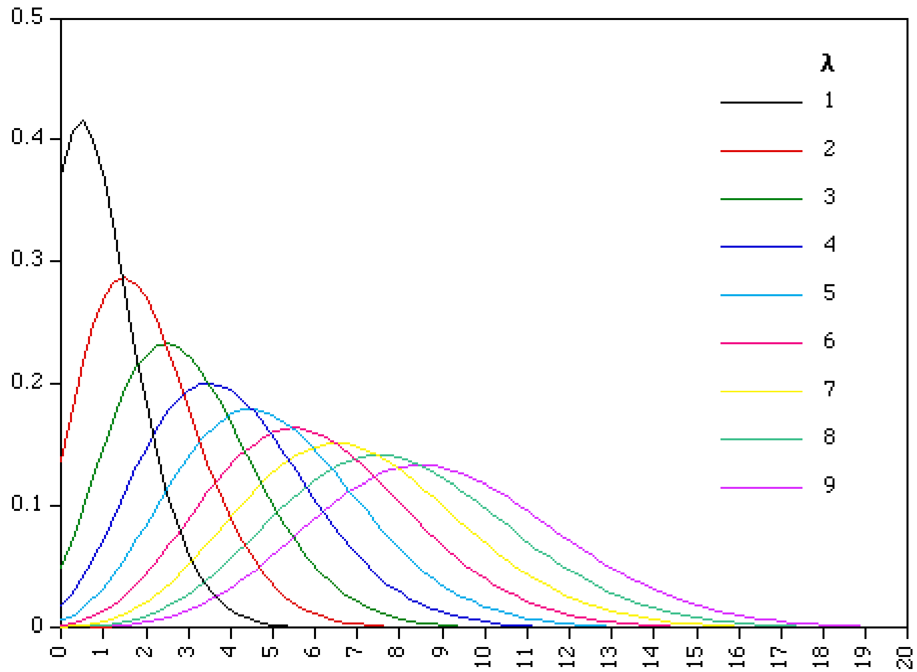
The Poisson Distribution

- like binomial, applies to variables taken on integer values > 0
- often used to model *counts* of events
  - number of phone calls placed in a given time period
  - number of times a neuron fires in a given time period
- single free parameter λ
- probability of exactly x events:
  - $\exp(-\lambda)\ \lambda^x/x!$
  - mean and variance are both λ
- binomial distribution with n large, $p = \lambda/n$ (λ fixed)
  - converges to Poisson with mean λ

The Poisson Distribution



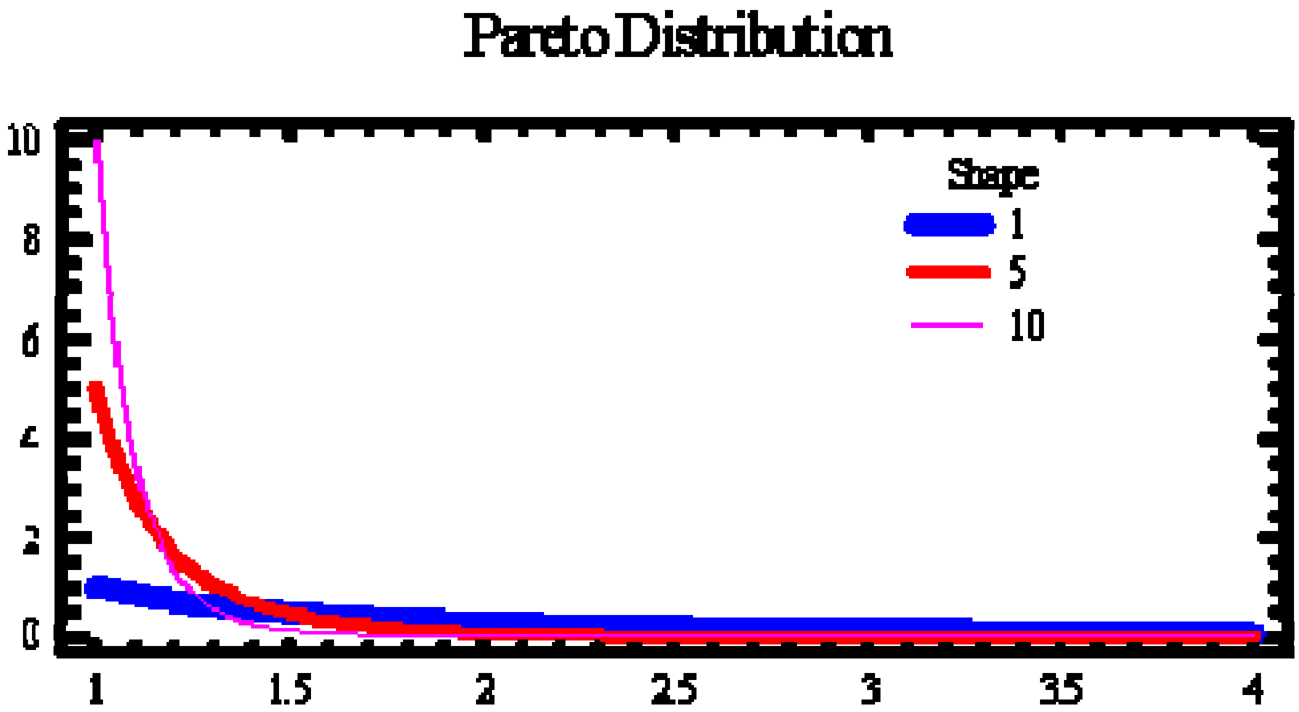
single photoelectron distribution



Heavy-tailed Distributions

- *Pareto* or *power law* distributions:
  - for variables assuming integer values > 0
  - probability of value x ~ 1/x^a
  - typically 0 < a < 2; smaller a gives heavier tail
  - sometimes also referred to as being *scale-free*
- For binomial, normal, and Poisson distributions the tail probabilities approach 0 *exponentially* fast
- Inverse *polynomial* decay vs. *inverse* exponential decay
- What kind of phenomena does this distribution model?
- What kind of process would *generate* it?

Heavy-Tailed Distributions



Distributions vs. Data

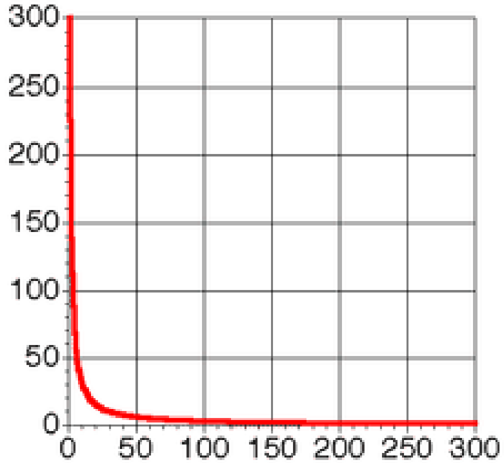
- All these distributions are *idealized models*
- In practice, we do not see distributions, but *data*
- Thus, there will be some *largest* value we observe
- Also, can be difficult to "eyeball" data and choose model
- So how do we distinguish between Poisson, power law, etc?
- Typical procedure:
  - might restrict our attention to a *range* of values of interest
  - accumulate *counts* of observed data into equal-sized bins
  - look at counts on a *log-log plot*
  - note that
    - power law:
      - $\log(\Pr[X = x]) = \log(1/x^{\alpha}) = -\alpha \log(x)$
      - linear, slope $-\alpha$
    - Normal:
      - $\log(\Pr[X = x]) = \log(a \exp(-x^2/b)) = \log(a) - x^2/b$
      - non-linear, concave near mean
    - Poisson:
      - $\log(\Pr[X = x]) = \log(\exp(-\lambda) \lambda^x/x!)$
      - also non-linear

Zipf's Law

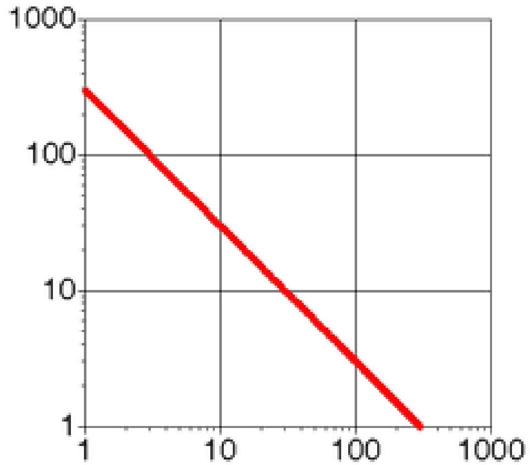
- Look at the frequency of English words:
  - "the" is the most common, followed by "of", "to", etc.
  - claim: frequency of the n-th most common ~ 1/n (power law, $\alpha = 1$)
- General theme:
  - *rank* events by their *frequency of occurrence*
  - resulting distribution often is a power law!
- Other examples:
  - North America city sizes
  - personal income
  - file sizes
  - genus sizes (number of species)
- People seem to dither over exact form of these distributions (e.g. value of α), but not heavy tails

Zipf's Law

The same data plotted on linear and logarithmic scales.
Both plots show a Zipf distribution with 300 datapoints



Linear scales on both axes



Logarithmic scales on both axes