Lower bound $\Omega(1/\varepsilon)$ words to get an ε-approximation
[use an argument based on the communication complexity: the INDEX problem]

WHAT IF $F[i]$ can be negative?

$$\tilde{F}[i] = \underset{1\le j\le r}{\text{median}}\ T(j, h_j(i))$$

[median has $\lfloor\frac{k-1}{2}\rfloor$ smaller elements]

FACT $F[i] - 3\varepsilon\|F\| \le \tilde{F}[i] \le F[i] + 3\varepsilon\|F\|$ with probability $\ge 1-\delta^{1/4}$
and $\|F\| = \sum_i |F[i]|$

Before proving this fact, we need to introduce

CHERNOFF'S BOUNDS (Motwani-Raghavan book '95)

$Y_1, Y_2, \dots, Y_r$ independent identically distributed (i.i.d) random variables
s.t. $\forall j: Pr[Y_j = 1] = p$ and $Pr[Y_j = 0] = q = 1-p$
Let $Y = \sum_{j=1}^{r} Y_j$ and $\mu = E[Y] = rp$. For any $\lambda > 0$

$$Pr[Y \ge (1+\lambda)\mu] < \left(\frac{e^\lambda}{(1+\lambda)^{1+\lambda}}\right)^\mu$$

We can now prove our fact, recalling that $\tilde{F}[i] = F[i] + X_{ji}$ for the chosen j

① Use Markov's inequality

$$Pr(|X_{ji}| > \underbrace{3\varepsilon\|F\|}_{a}) < \frac{E[|X_{ji}|]}{\underbrace{3\varepsilon\|F\|}_{a}} = \frac{\frac{\varepsilon}{e}\|F\|}{3\varepsilon\|F\|} = \frac{1}{3e} < \frac{1}{8}$$

② Use indicator variables

$$Y_j = \begin{cases} 1 & \text{if } |X_{ji}| > 3\varepsilon\|F\| \quad \text{with } p < \frac{1}{8} \\ 0 & \text{otherwise} \end{cases}$$

③ Observe that the median returns $T(j, h_j(i))$ with $|X_{ji}| \leq 3\varepsilon \|F\|$ when there are $< \frac{r}{2}$ cells $T(j', h_{j'}(i))$ with $|X_{j'i}| > 3\varepsilon \|F\|$

④ To estimate the probability of error, consider the event for which there are $\geq \frac{r}{2}$ such cells $T(j', h_{j'}(i))$: this is equivalent to

$$\boxed{Y \geq \frac{r}{2}}$$

⑤ Apply Chernoff's bounds and set $\mu = rp$ and $(1+\lambda)\mu = \frac{r}{2}$

$$\Pr[Y \geq (1+\lambda)\mu] < \left(\frac{e^{\lambda}}{(1+\lambda)^{1+\lambda}}\right)^{\mu} = \frac{1}{e^{\mu}}\left(\frac{e}{1+\lambda}\right)^{(1+\lambda)\mu}$$

⑥ We now bound $\frac{1}{e^{\mu}}\left(\frac{e}{1+\lambda}\right)^{(1+\lambda)\mu} = \frac{1}{e^{rp}}(2pe)^{\frac{r}{2}} \leq \frac{1}{2^{\frac{r}{4}}}$ $\leq \delta^{\frac{1}{4}}$

$$2^{\frac{r}{4}} \leq \underbrace{e^{rp}}_{\substack{\geq 1 \\ \text{as } rp \geq 0}} \frac{1}{(2pe)^{r/2}}$$

It suffices to have $\frac{1}{2pe} > \sqrt{2}$ iff $p < \frac{1}{2\sqrt{2}e}$, which is true since $p < \frac{1}{8}$ and $2\sqrt{2}e = 7.668\ldots$

Q.E.D.