Optimization methods to multicommodity flows

(Ahuja - Magnanti - Orlin : 17.4 - 17.5 - 17.6 - 17.7)

1 Lagrangian relaxation

(Ahuja - Magnanti - Orlin : 17.4)

Recall the general LP model:

$$z = \text{Min} \sum_{k=1}^{K} \sum_{(i,j)\in A} c_{ij}^k x_{ij}^k$$

(MCF 1)

$$\sum_{(i,j)\in FS(i)} x_{ij}^k - \sum_{(j,i)\in BS(i)} x_{ji}^k = b_i^k \quad \forall i \in N, \ k = 1, \dots K$$

$$\boxed{\sum_{k=1}^{K} x_{ij}^k \le u_{ij} \quad \forall (i,j)\in A} \quad \pi_{ij}$$

$$x_{ij}^k \geq 0 \quad \forall (i,j)\in A, \ k = 1, \dots K$$

where:

K : number of commodities

x_{ij}^k : amount of flow pushed along (i,j) for commodity k

To apply Lagrangian relaxation, associate a nonnegative Lagrangian multiplier π_{ij} with each bundle constraint:

$$z(\pi) = \min \sum_{k=1}^{K} \sum_{(i,j)\in A} (c_{ij}^k + \pi_{ij})\, x_{ij}^k - \sum_{(i,j)\in A} \pi_{ij} u_{ij}$$

$$\sum_{(i,j)\in FS(i)} x_{ij}^k - \sum_{(j,i)\in BS(i)} x_{ji}^k = b_i^k \quad \forall i \in N, \quad k = 1, \dots K$$

$$x_{ij}^k \geq 0 \qquad \forall (i,j)\in A, \; k = 1, \dots K$$

For given $\{\pi_{ij}\}$ the term $\sum_{(i,j)\in A} \pi_{ij} u_{ij}$ is constant. Therefore, the resulting problem decomposes into separate minimum cost flow problems, one for each commodity, with costs $\{c_{ij}^k + \pi_{ij}\}$.

Note that, since (MCF 1) is a linear program, if we solve the corresponding Lagrangian Dual we get

$$Z = \max_{\pi : \pi_{ij} \geq 0} z(\pi)$$

For example, if we use the Subgradient algorithm to solve the Lagrangian Dual, then at each step we can update the Lagrangian multipliers according to:

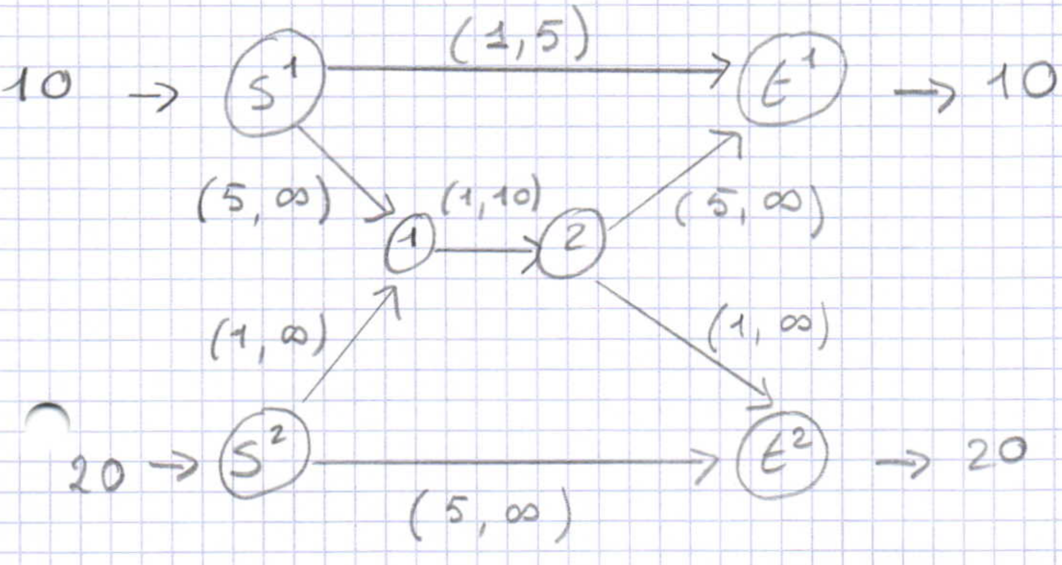
$$\pi_{ij}^{t+1} = \max\left\{ \pi_{ij}^{t} + \underbrace{\lambda_t}_{\text{step size}} \left(\sum_{k=1}^{K} x_{ij}^{*k} - u_{ij} \right), 0 \right\},$$

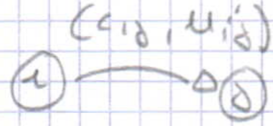
where:

- t current iteration
- $\{x_{ij}^{*k}\}$ optimal solution for commodity k

Recall, in fact, that $\left(\sum_{k=1}^{K} x_{ij}^{*k} - u_{ij} \right)$ is a subgradient of the Lagrangian function at $\{\pi_{ij}^{t}\}$.

<u>example</u> (pages 661-662-663) 659





$c_{ij}^{k} = c_{ij}$

$k = 1, 2$