# INTRODUCTION TO NETWORK SCIENCE

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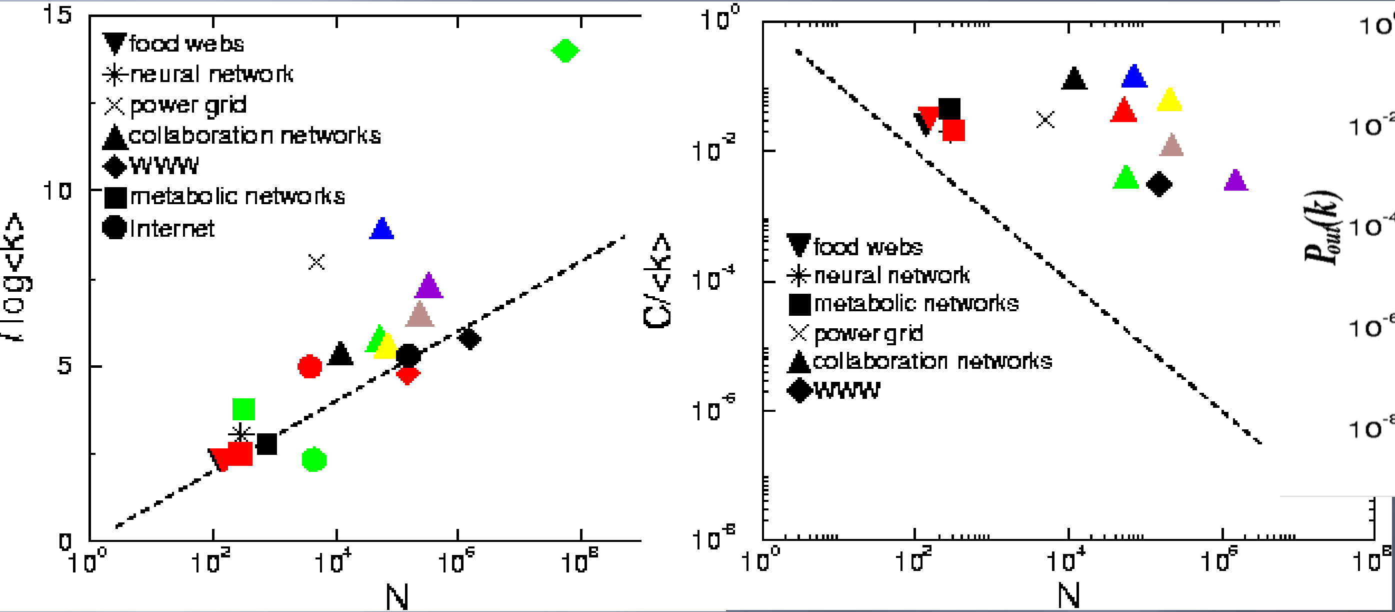
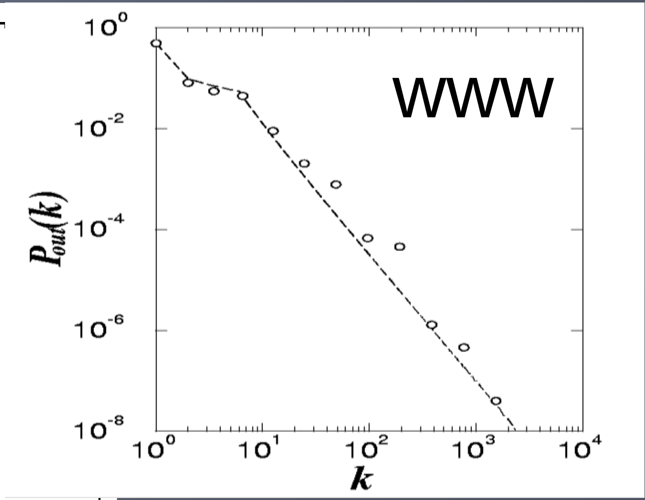
## 6. NETWORK GROWTH MODELS

# Complex networks: observations




$\langle d \rangle \sim \log N$

$C / \langle k \rangle = const$

$p_k \sim k^{-\gamma}$

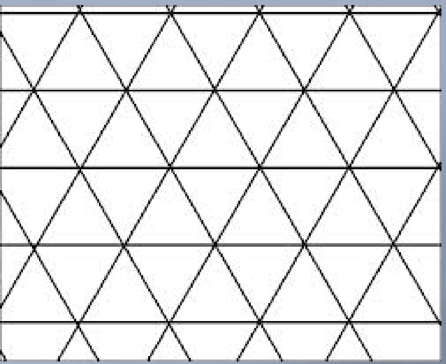
**Small World:**
Average distance scales logarithmically with the network size

**Clustered**:
Clustering coefficient is large, it does not depend on network size.

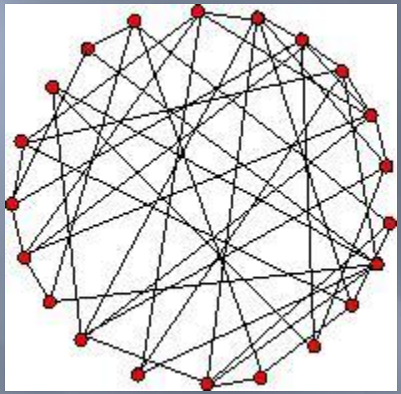
**Scale-free:**
The degree distribution is broad, with a power law tail.

# Complex networks: static models

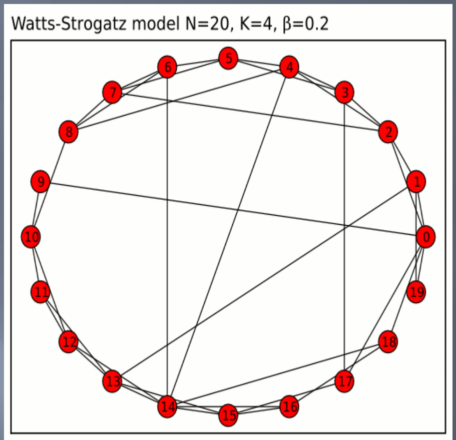
Lattice



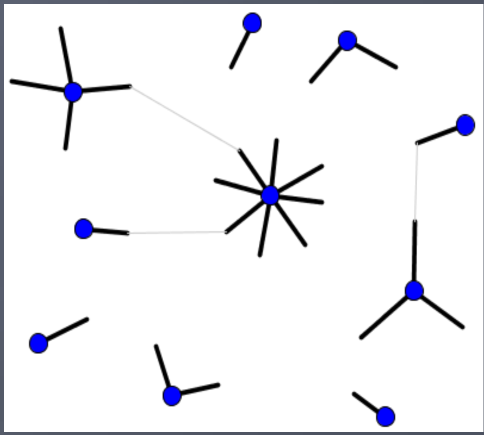
ER graph



WS graph




Configuration



# Tayloring vs Modeling

With configuration type models all requested features can be described – no model in the deeper sense

Amazing universal behavior across a large number of networks: small world, high clustering, broad degree distribution (qualitative universality).

Universality: common mechanism for emergence?

What is common in collaboration, internet, genetic etc. networks?!

# Problem of modeling

Modeling: matter of culture

For a statistician: Modeling is to find a function, which fits data best.

The more we learn about our system the more sophisticated models of this kind are needed.

Such models can be of practical use: You don't need to memorize all the details, you can take the model.

It is like a map of a country. However, if you want more details the resolution of your map has to be better (the fitting functions will have more parameters).

# Problem of modeling

Folklore in physics:

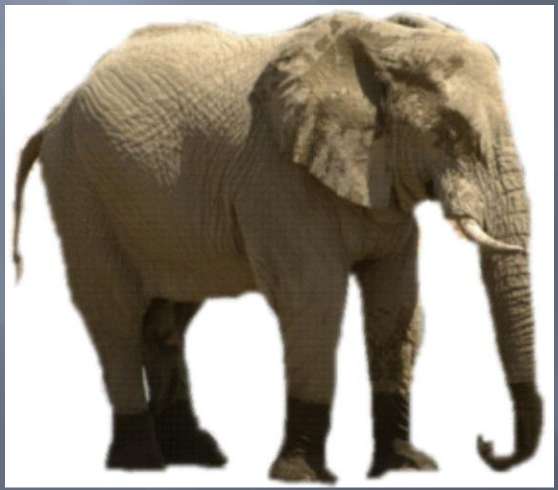
With 2 parameters

Straight line

With 3 parameters

Parabola

With 4 parameters



With parameters > 4

# Problem of modeling

"What do you consider the largest map that would be really useful?"
"About six inches to the mile."
"Only six inches!" exclaimed Mein Herr. "We very soon got six yards to the mile. Then we tried a hundred yards to the mile. And then came the grandest idea of all! We actually made a map of the country, on the scale of a mile to the mile!"
"Have you used it much?" I enquired.
"It has never been spread out, yet," said Mein Herr: "The farmers objected: they said it would cover the whole country, and shut out the sunlight! So now we use the country itself, as its own map, and I assure you it does nearly as well."

Lewis Carroll's Sylvie and Bruno Concluded

# Problem of modeling

Without any theoretical support, such models (fitting functions, maps) do not help much in understanding the phenomenon.

We require from a model more!
It should give insight into the basic mechanisms.

Kepler’s 3 laws: Description

Newton’s laws + gravity law: Cause → consequence.

From Newton-type laws there is hope to draw conclusions beyond the observations, predictions e.g., discovery of new planets.

# Problem of modeling

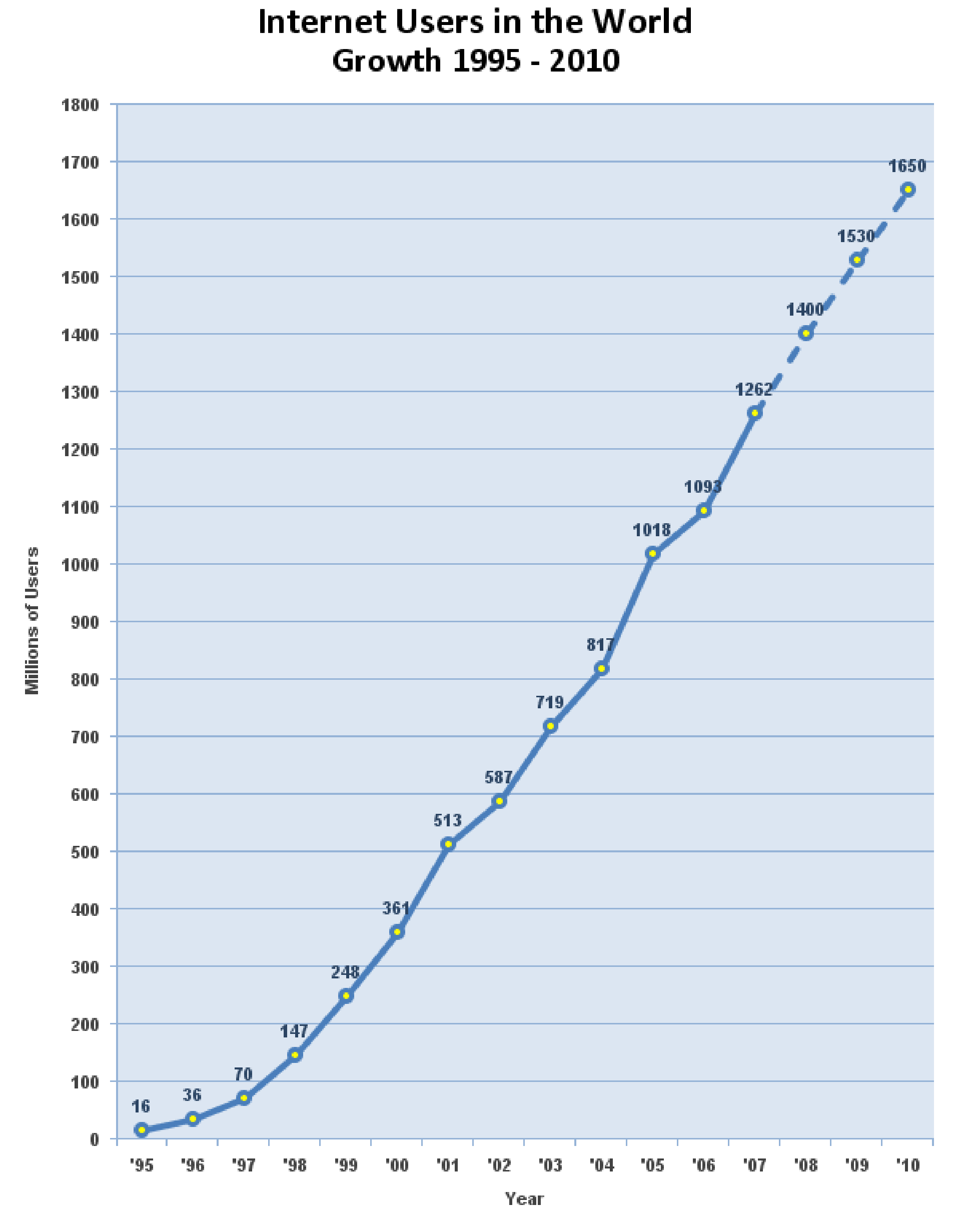
We need Newton's laws for networks.

It is not enough to study complex networks as they are given. We have to study how they emerge!

Most networks result from a growth process.

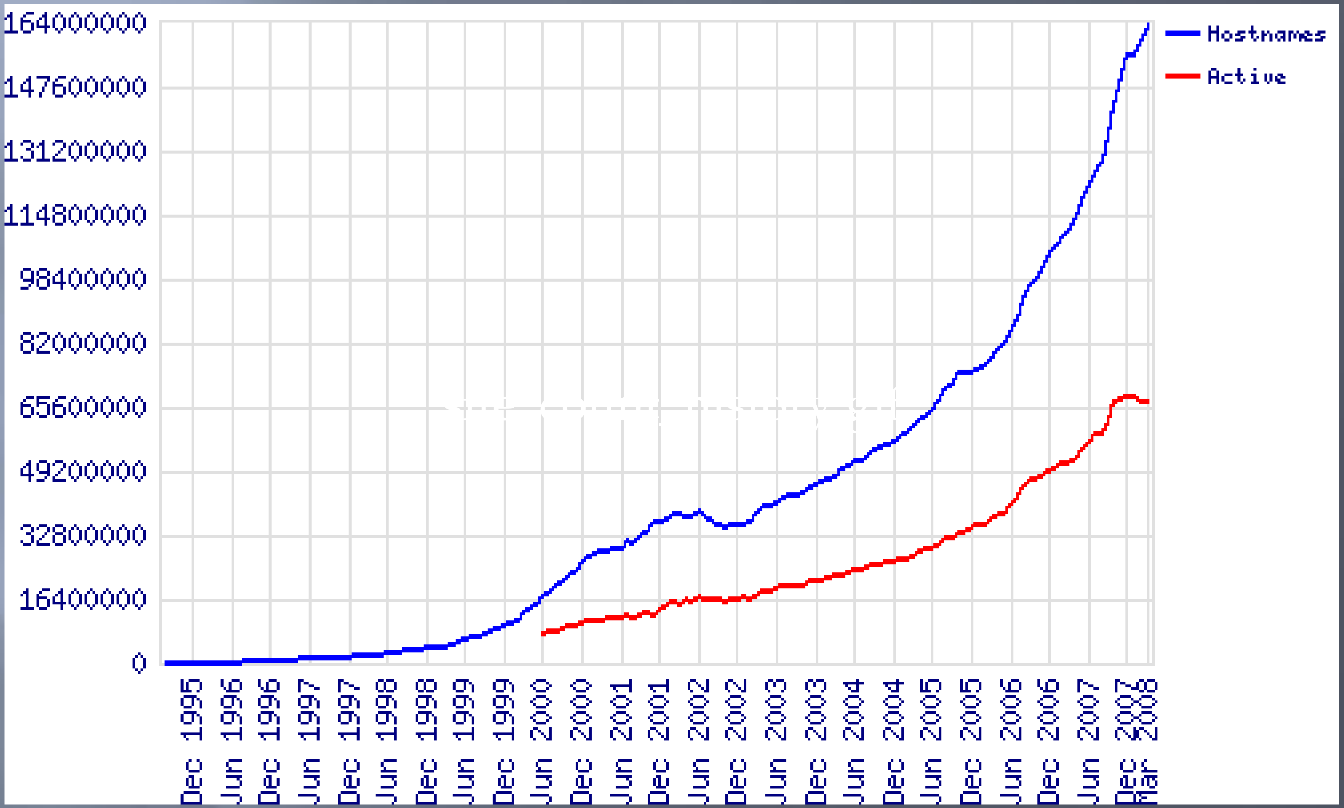
Random networks have constant number of nodes – they are unable to capture the growth aspect.

# Growing networks



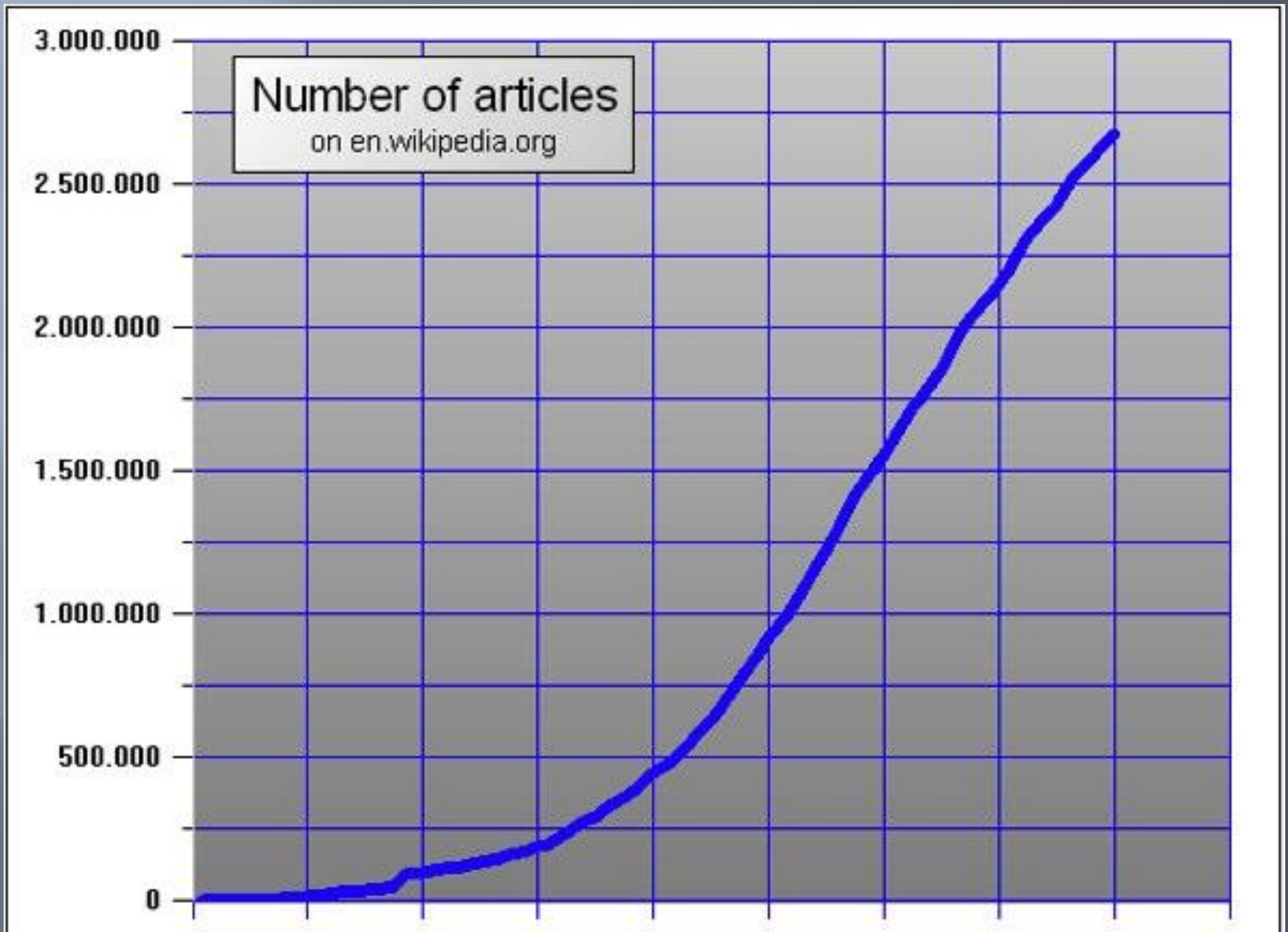

Source: www.internetworldstats.com - January, 2008

# Growing networks

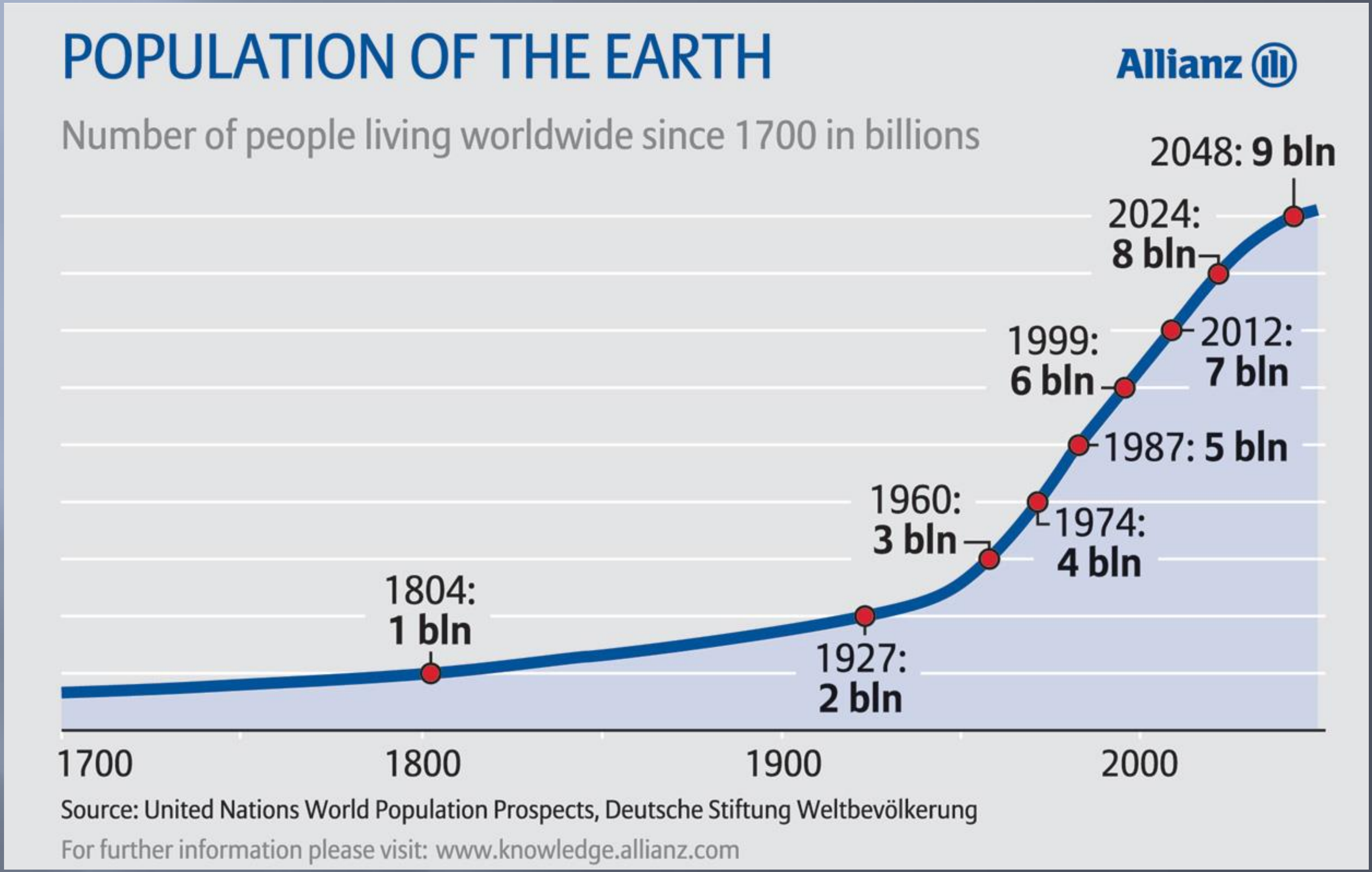



site_count_history.gif

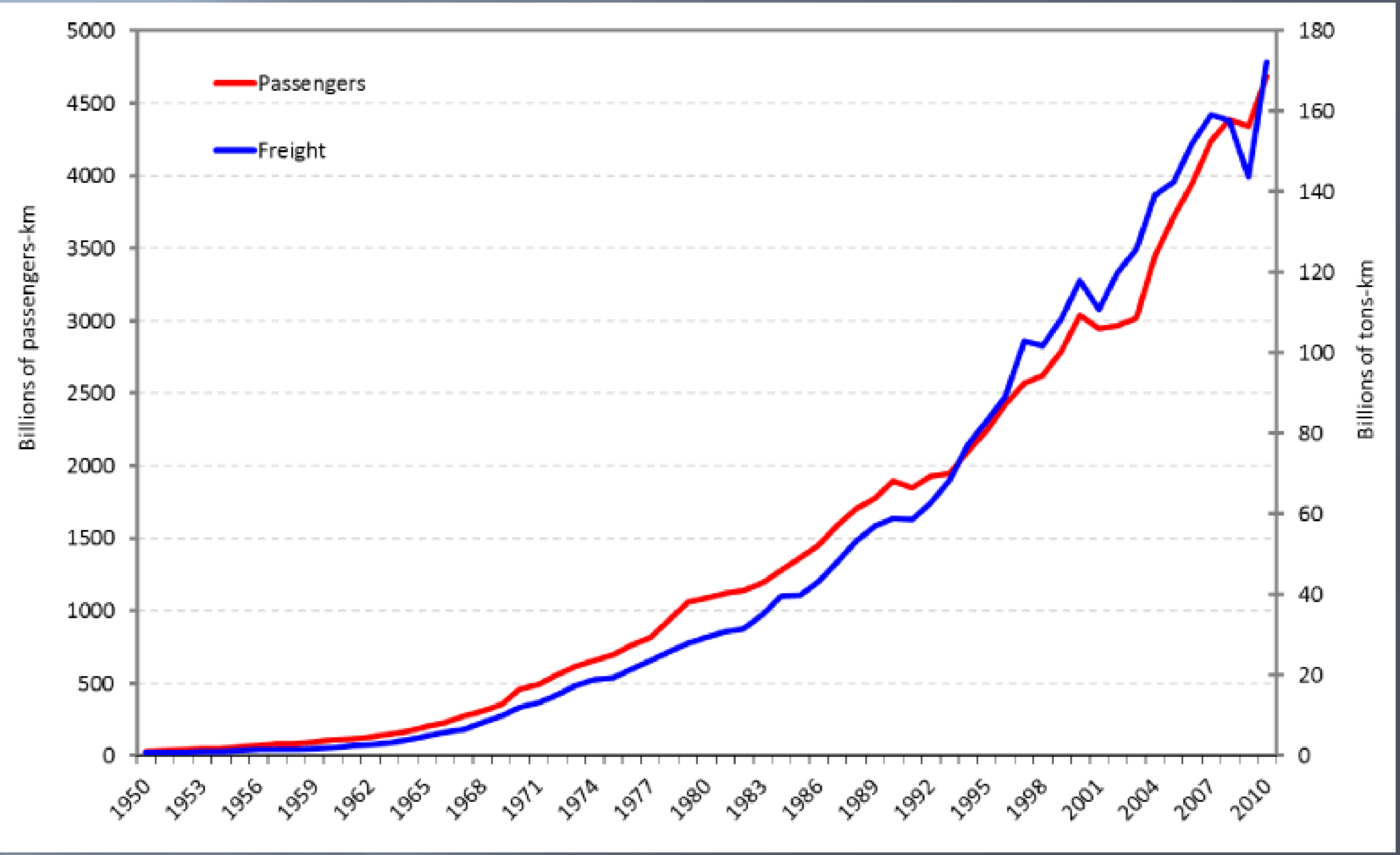
# Growing networks



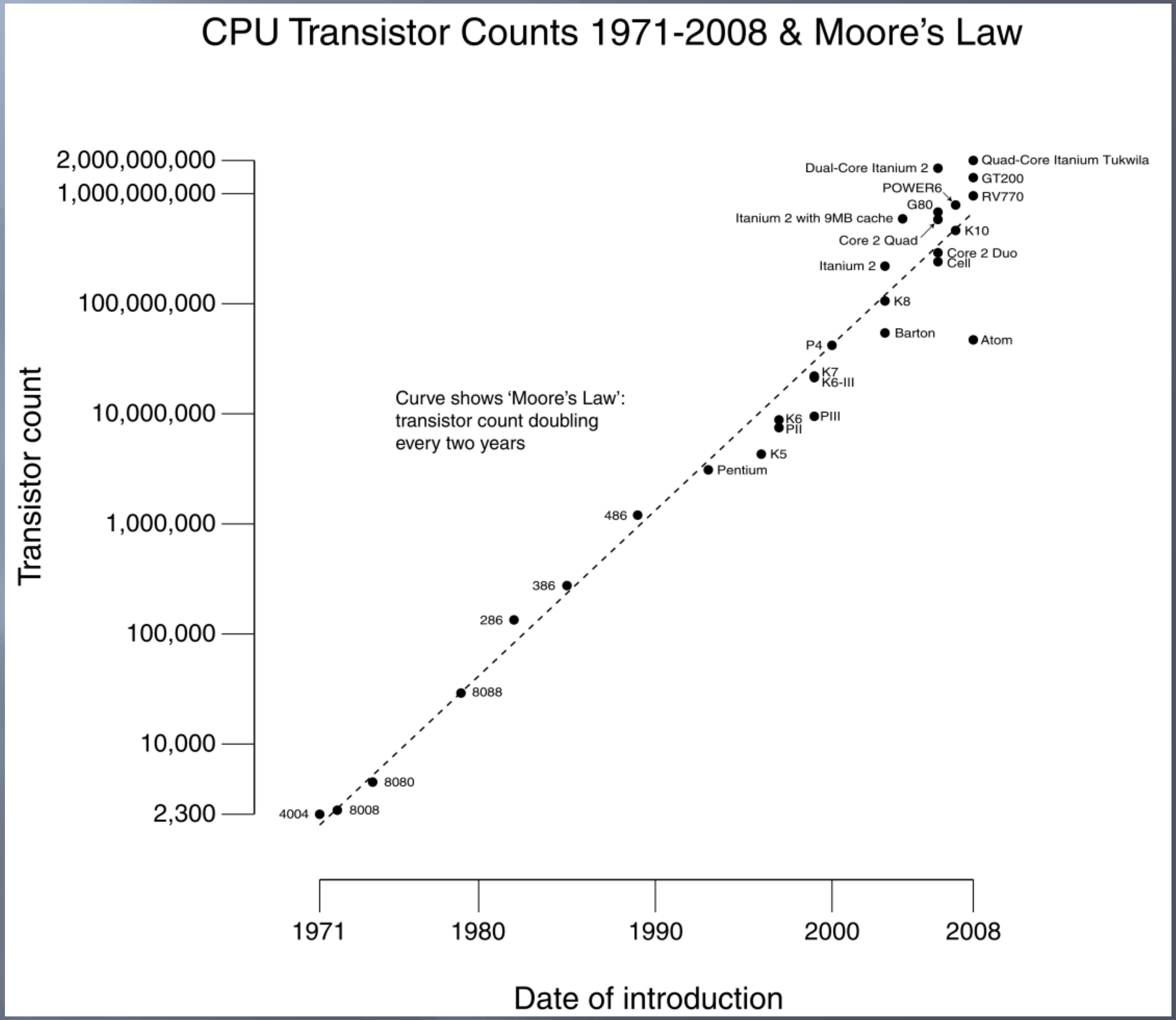
# Growing networks



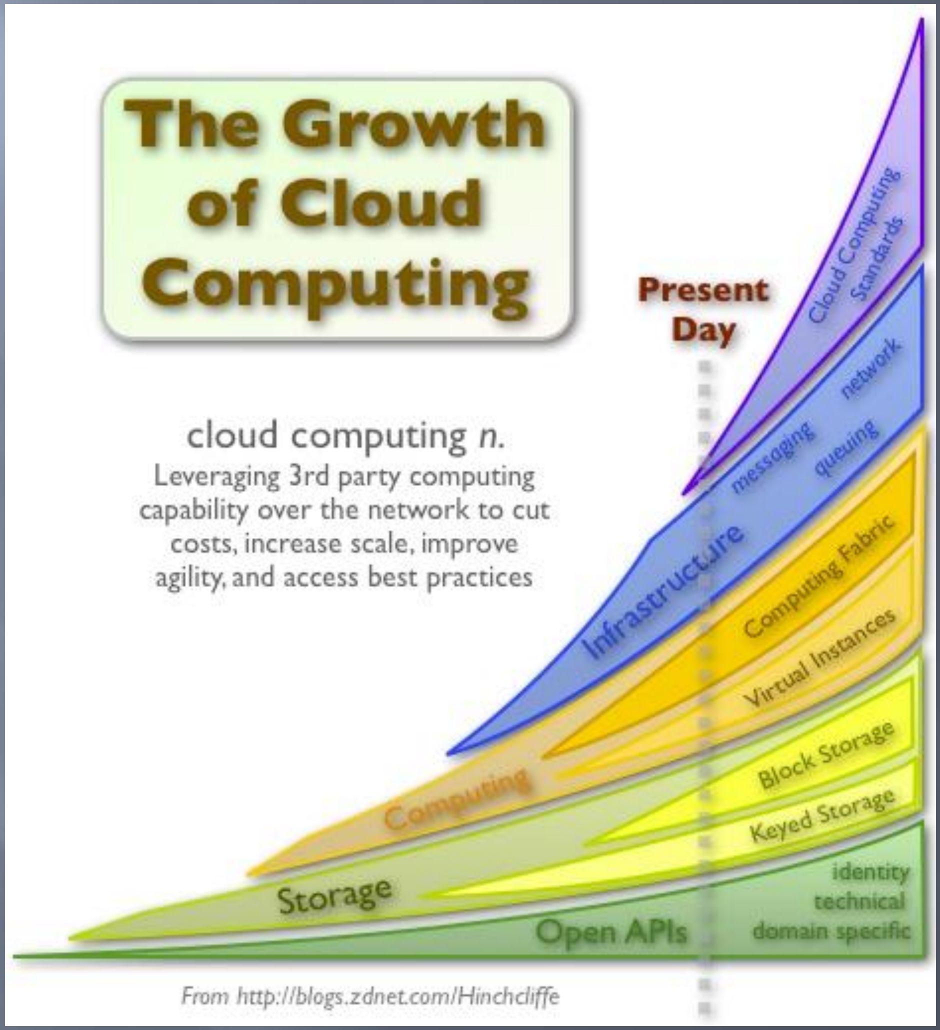
# Growing networks



# Growing networks



# Growing networks



# Growth

The simplest case is if we add nodes one by one and assume that all of them bring in *m* links.

The process needs a „seed”, i.e., an initial small set of nodes linked together.

$m = 2$

Next question: how to attach?

# Preferential attachment

The spirit of ER or WS: attach entirely randomly!

Wrong! It does not lead to the required broad degree distribution.

Attachment is not purely random. There are hubs indicating that the Matthew effect is in play

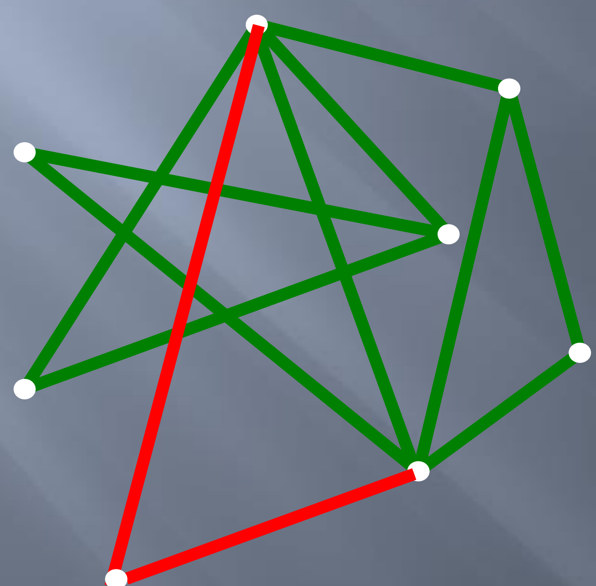
# Barabási-Albert (BA) model

Combination of
- Growth
- Preferential attachment

New nodes prefer to link to highly connected nodes.

László Barabási

Réka Albert



**PREFERENTIAL ATTACHMENT:**

The probability that a node connects to a node with *k* links is proportional to *k*.

$$\Pi(i) = \frac{k_i}{\Sigma_j k_j}$$

Normalization

Barabási & Albert, *Science* **286,** 509 (1999)

# Barabási-Albert (BA) model

(1) Networks continuously expand by the addition of new nodes

WWW : addition of new documents

**GROWTH**

(2) New nodes prefer to link to highly connected nodes.

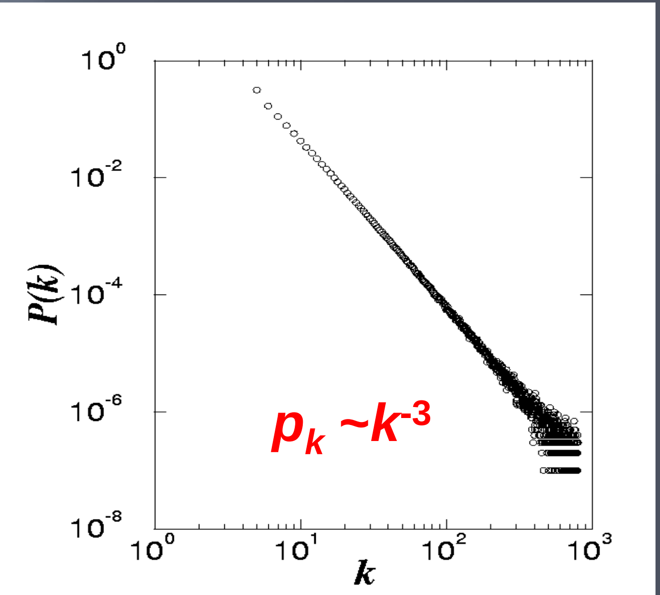
WWW : linking to well known sites

**PREFERENTIAL ATTACHMENT**

What comes out?

$$\Pi(k_i) = \frac{k_i}{\Sigma_j k_j}$$

Most interesting:
Degree distribution




Power law distribution of degrees

$\gamma$=3 independent of $m$

# Barabási-Albert (BA) model

Power law out of preferential attachment is not that new:

Gyorgy Polya (1887-1985) 1923: *Polya process* in the mathematics literature
George Udmy Yule (1871-1951) in 1925: the number of species per genus of flowering plants; *Yule process* in statistics
Robert Gibrat (1904-1980), 1931: rule of proportional growth independent of system size. *Gibrat process* in economics
George Kinsley Zipf (1902-1950), 1949: the distribution of wealth in the society.
Herbert Alexander Simon (1916-2001), 1955, the distribution of city sizes and other phenomena
Derek de Solla Price (1922-1983), 1976, used it to explain the citation statistics of scientific publications, "cumulative advantage"
Robert Merton (1910-2003), 1968: *Matthew effect*,

B A: simple network model
+ the ubiquity of preferential attachment for networks resulted in a radically new approach to modelling them

# Barabási-Albert (BA) model

Derivation [*N(t) = t;* one node per time step]*:*

$$< N(k,t) >= tp_k(t)$$ Number of nodes with degree *k* at time *t*.

$$\Pi(k) = \frac{k}{\sum_j k_j} = \frac{k}{2mt}$$

*2m*: each node adds *m* links, but each link contributes to the degree of 2 nodes

Number of links added to degree *k* nodes after the arrival of a new node:

Total number of k-nodes

$$\frac{k}{2mt} \times tp_k(t) \times m = \frac{k}{2} p_k(t)$$

Preferential attachment

New node adds m new links

# of degree *k-1* nodes that acquire a new link, becoming degree *k*

$$\frac{k-1}{2} p_{k-1}(t)$$

# of degree *k* nodes that acquire a new link, becoming degree *k+1*

$$\frac{k}{2} p_k(t)$$

$$(t+1) p_k(t+1) = tp_k(t) + \frac{k-1}{2} p_{k-1}(t) - \frac{k}{2} p_k(t)$$

# k-nodes at time t+1

# k-nodes at time t

Gain of k-nodes via k-1→ k

Loss of k-nodes via k→ k+1

A.-L.Barabási, R. Albert and H. Jeong, *Physica A* **272**, 173 (1999)

# Barabási-Albert (BA) model

$$(t+1)p_k(t+1) = tp_k(t) + \frac{k-1}{2}p_{k-1}(t) - \frac{k}{2}p_k(t)$$

No nodes with degree < *m*. We need a separate eq. for that case:

$$(t+1)p_m(t+1) = tp_m(t) + \underbrace{1}_{\text{The just arriving new node}} - \frac{m}{2}p_m(t)$$

The just arriving new node

We are interested in the long time, stationary solution:

$$\lim_{t\to\infty} p_k(t) = p_k$$

# Barabási-Albert (BA) model

$$(t+1)\,p_k(t+1) = tp_k(t) + \frac{k-1}{2}\,p_{k-1}(t) - \frac{k}{2}\,p_k(t) \quad k > m$$

$$(t+1)\,p_m(t+1) = tp_m(t) + \quad 1 \quad - \frac{m}{2}\,p_m(t)$$

Stationary equations with

$$\lim_{t\to\infty} p_k(t) = p_k$$

$$p_k = \frac{k-1}{2}\,p_{k-1} - \frac{k}{2}\,p_k \quad k > m$$

$$p_k = \frac{k-1}{k+2}\,p_{k-1} \quad k > m$$

$$p_m = 1 - \frac{m}{2}\,p_m$$

$$p_m = \frac{2}{m+2}$$

# Barabási-Albert (BA) model

$$p_k = \frac{k-1}{k+2} p_{k-1}$$ ➔ $$p_{k+1} = \frac{k}{k+3} p_k$$

$$p_m = \frac{2}{m+2}$$

„Initial condition”

$$p_{m+1} = \frac{m}{m+3} p_m = \frac{2m}{(m+2)(m+3)} = \frac{2m(m+1)}{(m+1)(m+2)(m+3)}$$

$$p_{m+2} = \frac{m+1}{m+4} p_{m+1} = \frac{2m(m+1)}{(m+2)(m+3)(m+4)}$$

$$p_{m+3} = \frac{m+2}{m+5} p_{m+2} = \frac{2m(m+1)}{(m+3)(m+4)(m+5)}$$

…

$$p_k = \frac{2m(m+1)}{k(k+1)(k+2)}$$

For large *k*: $P(k) \sim k^{-3}$

Power law tail

Krapivsky, Redner, Leyvraz, PRL 2000
Dorogovtsev, Mendes, Samukhin, PRL 2000
Bollobas et al, Random Struc. Alg. 2001

# A simple route

Start from eq. $$P(k) = \frac{k-1}{2} P(k-1) - \frac{k}{2} P(k)$$

$$2P(k) = (k-1)P(k-1) - kP(k) = -P(k-1) - k[P(k) - P(k-1)]$$

Let's take a continuum limit

$$2P(k) = -P(k-1) - k\frac{P(k) - P(k-1)}{k-(k-1)} = -P(k-1) - k\frac{dP(k)}{dk}$$

$$P(k) = -\frac{1}{2}\frac{d[kP(k)]}{dk}$$

$$P(k) = -\frac{1}{2}P(k) - \frac{1}{2}k\frac{dP(k)}{dk}$$

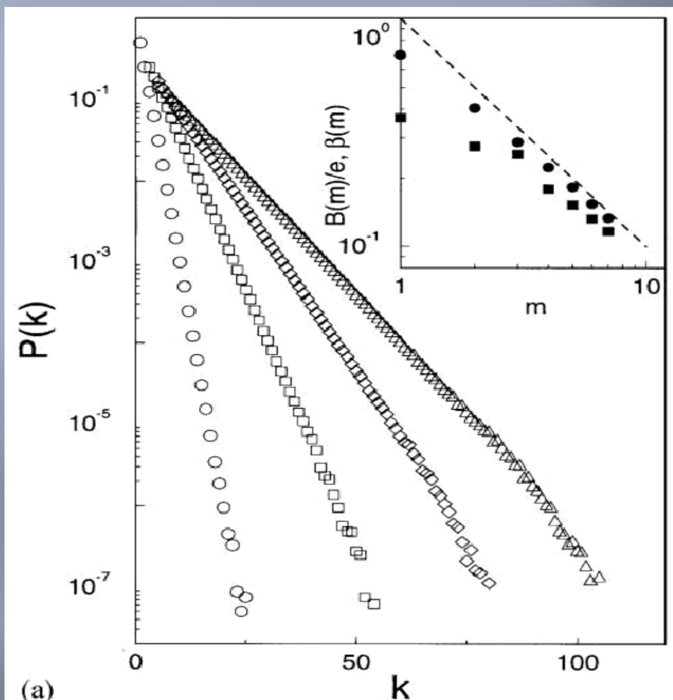
$$\frac{3}{2}P(k) = -\frac{1}{2}k\frac{dP(k)}{dk}$$

$$P(k) = Ak^{-3}$$

because $$y = ax\frac{dy}{dx} \Rightarrow y = Cx^{1/a}$$

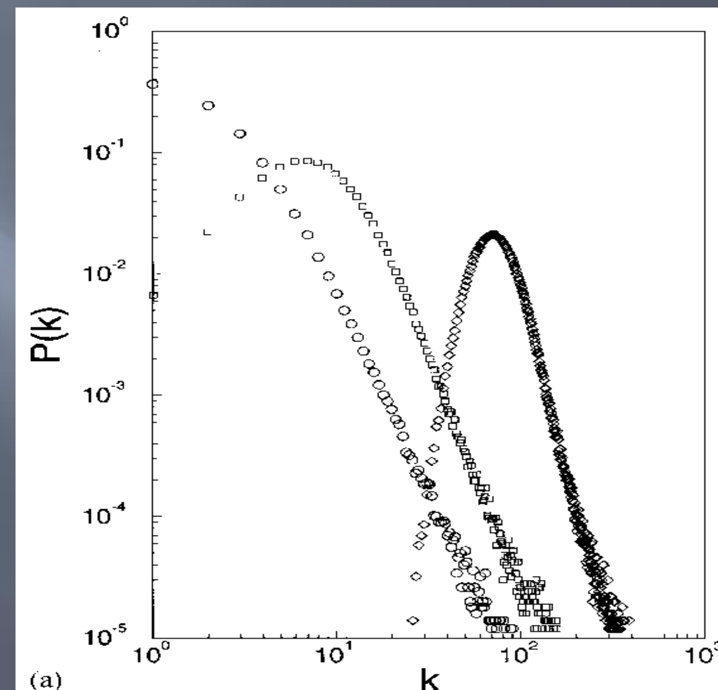
# Barabási-Albert (BA) model

Growth without P.A.




Exponential $p_k$

P.A. without growth ($N$ fixed)




Power law only at the beginning, then sharply peaked $p_k$ finally complete graph. No stationary sol'n

# Barabási-Albert (BA) model

| | Lattice | ER | WS | BA |
|---|---|---|---|---|
| $\langle d \rangle$ | $\sim L$ | $\sim \ln N$ | $\sim \ln N$ | |
| $C$ | const | $\langle k \rangle / N$ | const | |
| $p_k$ | $\delta(k,k_0)$ | Poisson | shifted Poisson | $\sim k^{-\gamma}$ |

Many questions:
Average distance $\langle d \rangle$?
Clustering $C$?
What about exponents other than 3?
Origin of preferential attachment (global rule)?

# Barabási-Albert (BA) model

$$\langle d \rangle \sim \begin{cases} const. & \gamma = 2 \\ \dfrac{\ln \ln N}{\ln(\gamma - 1)} & 2 < \gamma < 3 \\ \dfrac{\ln N}{\ln \ln N} & \gamma = 3 \\ \ln N & \gamma > 3 \end{cases}$$

Ultra Small World

Small World

Size of the biggest hub $O(N)$

Av. distance increases slower than log

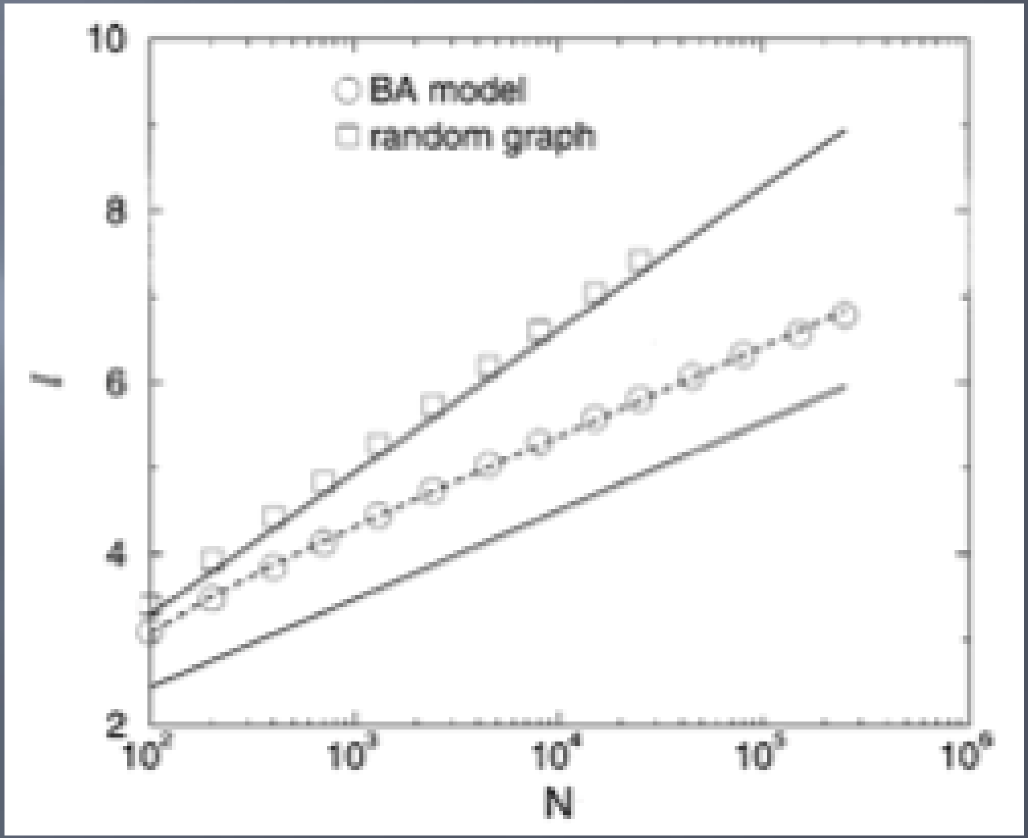
BA: Boarderline case: slightly slower than log

Finite second moment, result like in ER.

Config. model results but broader validity

# Barabási-Albert (BA) model

$$\langle d \rangle \approx \frac{\ln N}{\ln \ln N}$$




Bollobas, Riordan, 2002

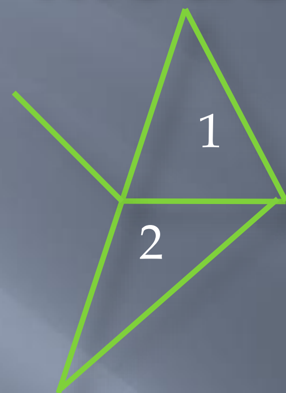
# Barabási-Albert (BA) model

| | Lattice | ER | WS | BA |
|---|---|---|---|---|
| $\langle d \rangle$ | $\sim L$ | $\sim \ln N$ | $\sim \ln N$ | $\sim \ln N / \ln\ln N$ |
| $C$ | const | $\langle k \rangle / N$ | const | |
| $p_k$ | $\delta(k,k_0)$ | Poisson | shifted Poisson | $\sim k^{-\gamma}$ |

BA model is a small world. The mechanism is through the hubs!  (C.f. Milgram experiment!)

# Clustering Coefficient in the BA model

$$C_i = \frac{N_i^{\Delta}}{k_i(k_i - 1)/2}$$




$$C = \frac{2}{6}$$

Denote the probability to have a link between node *i* and *j* with *P(i,j)* The probability that three nodes *i,j,l* form a triangle is *P(i,j)P(i,l)P(j,l).* The expected number of triangles, in which a node $l$ with degree $k_l$ participates is thus:

$$N_l^{\Delta} = \int_{i=1}^{N} di \int_{j=1}^{N} dj P(i,j) P(i,l) P(j,l)$$

We need to calculate P(i,j).

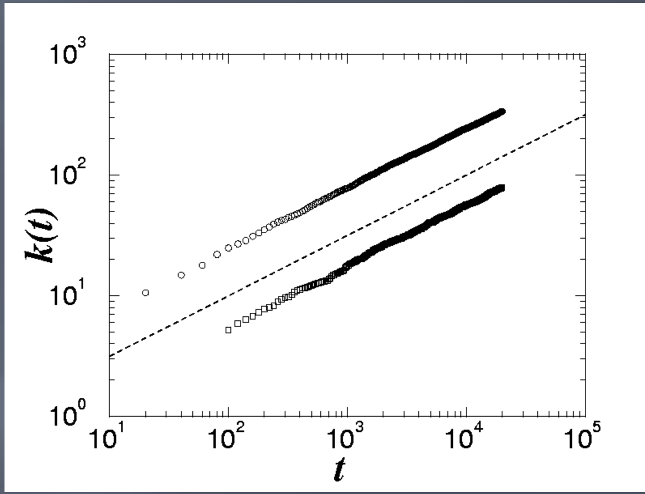
# Time evolution of k

Let us describe the time evolution of degrees

$$k_i(t+1) = k_i(t) + m\frac{k_i}{\sum_j k_j} \Rightarrow \frac{dk_i}{dt} = m\frac{k_i}{\sum_j k_j} = \frac{k_i}{2t}$$

$$\frac{dk_i}{dt} = \frac{k_i}{2t}$$

$$k_i(t) = m\sqrt{\frac{t}{t_i}} \sim t^{\beta} \quad \text{with } \beta = 1/2$$




This tells us that what matters is the age of the nodes. Old nodes will always have advantage!

A.-L.Barabási, R. Albert and H. Jeong, *Physica* A **272,** 173 (1999)

# Calculate P(i,j)

Node $j$ arrives at time $t_j=j$ and the probability that it will link to node $i$ with degree $k_i$ already in the network is determined by preferential attachment:

time $j$

$$P(i,j) = m\Pi(k_i(j)) = m\frac{k_i(j)}{\sum_{l=1}^{j} k_l} = m\frac{k_i(j)}{2mj}$$

$$k_i(t) = m\left(\frac{t}{t_i}\right)^{1/2} = m\left(\frac{j}{i}\right)^{1/2}$$

Where we used that the arrival time of node $j$ is $t_j=j$ and the arrival time of node is $t_i=i$

$$P(i,j) = \frac{m}{2}(ij)^{-\frac{1}{2}}$$

$$N_i^{\Delta} = \int_{i=1}^{N} di \int_{j=1}^{N} dj P(i,j)P(i,l)P(j,l) = \frac{m^3}{8}\int_{i=1}^{N} di \int_{j=1}^{N} dj (ij)^{-\frac{1}{2}}(il)^{-\frac{1}{2}}(jl)^{-\frac{1}{2}} = \frac{m^3}{8l}\int_{i=1}^{N}\frac{di}{i}\int_{j=1}^{N}\frac{dj}{j} = \frac{m^3}{8l}(\ln N)^2$$

$$C = \frac{\frac{m^3}{8l}(\ln N)^2}{k_l(k_l - 1)/2}$$

$$k_l(t) = m\left(\frac{N}{l}\right)^{1/2}$$

Which is the degree of node $i$ at current time, at time $t=N$

Let us approximate:

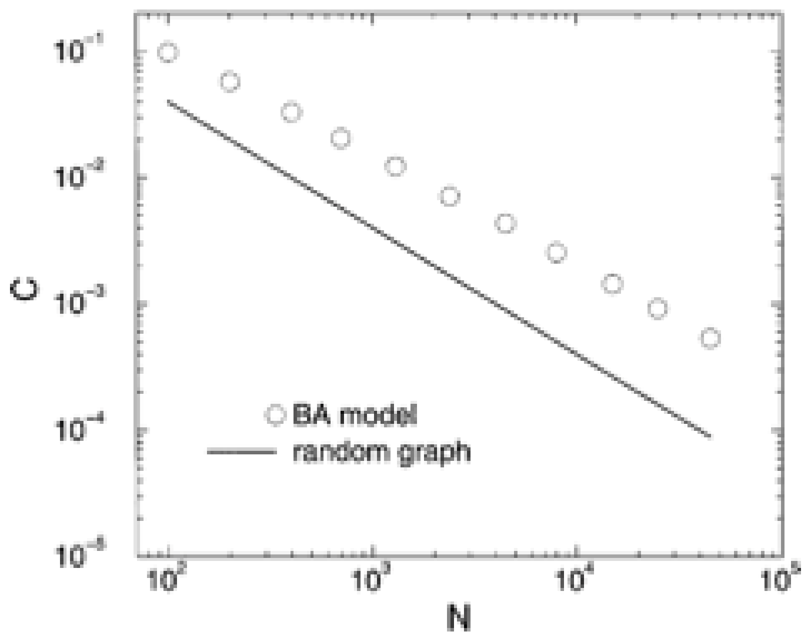
$$k_l(k_l - 1) \approx k_l^2 = m^2\frac{N}{l}$$

$$C = \frac{m}{8}\frac{(\ln N)^2}{N}$$

# Barabási-Albert (BA) model

Clustering coefficient:

For ER we have $C = p = \langle k \rangle / N$
Decreasing with size. BA:




Bad news!

$$C = \frac{m}{8} \frac{(\ln N)^2}{N}$$

Konstantin Klemm, Victor M. Eguiluz, 2002

# Barabási-Albert (BA) model

| | Lattice | ER | WS | BA |
|---|---|---|---|---|
| $<d>$ | $\sim L$ | $\sim \ln N$ | $\sim \ln N$ | $\sim \ln N / \ln\ln N$ |
| $C$ | const | $<k>/N$ | const | $\sim(\ln N)^2 / N$ |
| $p_k$ | $\delta(k,k_C)$ | Poisson | shifted Poisson | $\sim k^{-\gamma}$ |

Shall we throw BA away? No!

It is an important aspect to capture the mechanism by which complex networks emerge. A new approach, a starting point.

# Barabási-Albert (BA) model

## Summary of the BA model:

- **Nr. of nodes:** $N = t$
- **Nr. of links:** $L = m\,t$
- **Average degree:** $\langle k \rangle = \frac{2L}{N} \to 2m$
- **Degree dynamics** $k_i(t) = m\left(\frac{t}{t_i}\right)^{\beta} \quad \beta = \frac{1}{2}$ β: dynamical exponent
- **Degree distribution:** $P(k) \sim k^{-g} \quad g = 3$ γ: degree exponent
- **Average Path Length:** $\boldsymbol{l} \approx \frac{\ln \boldsymbol{N}}{\ln \ln \boldsymbol{N}}$
- **Clustering Coefficient:** $C \sim \frac{(\ln N)^2}{N}$

The network grows, but the degree distribution becomes stationary.

# Further network growth models

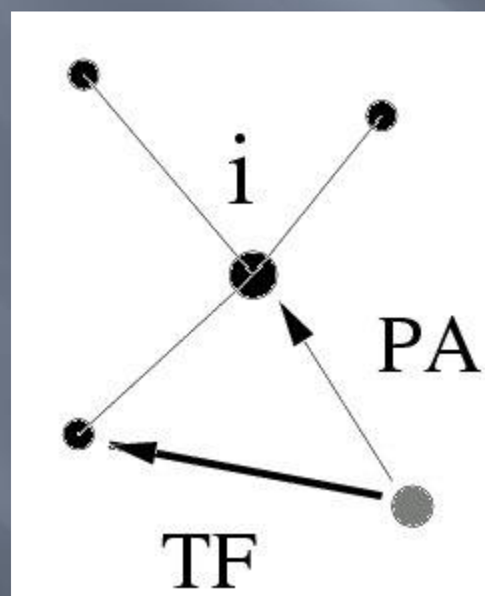
How to cure the clustering problem?

Something is missing from the attachment mechanism

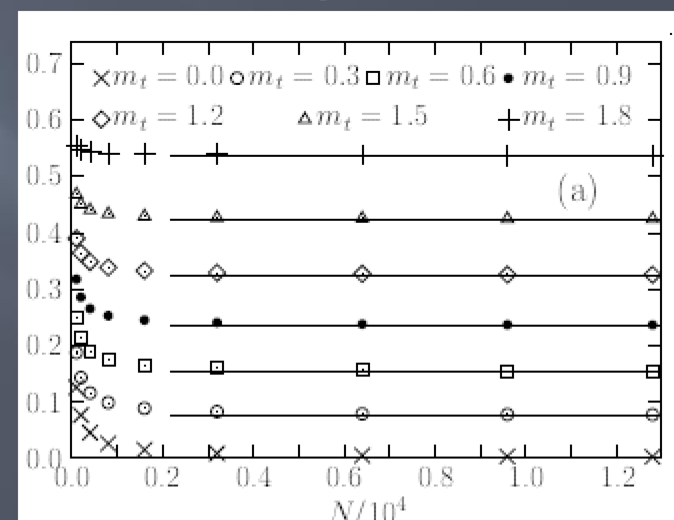
Besides preferential attachment (hunting for popular nodes), there is an additional process: Friends of friends get easily friends. This should be incorporated.

PA: Preferencial attachment
TF: triad formation

First link: PA
then TF with prob. $p$
PA with $1\text{-}p$




C




P. Holme and B.J. Kim 2002

# Further network growth models

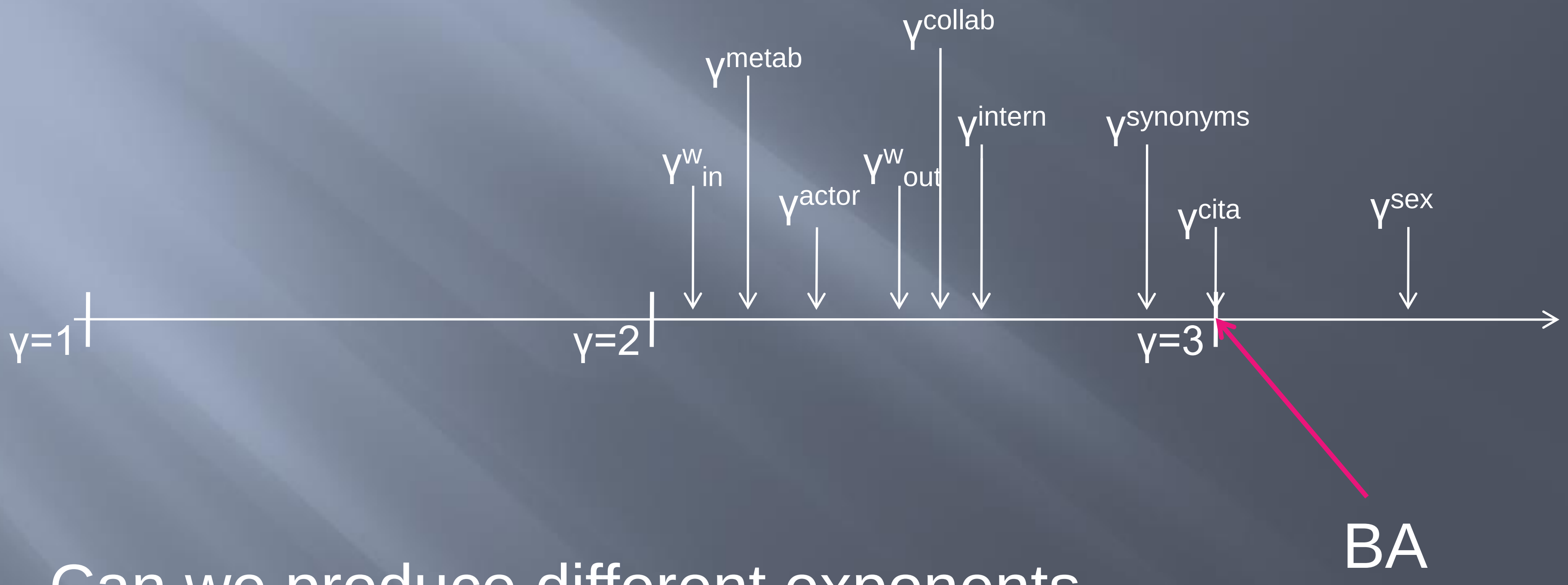
This was easy! (Much easier than the effort with the configuration model!)

| | Lattice | ER | WS | BA+ |
|---|---|---|---|---|
| $\langle d \rangle$ | $\sim L$ 👎 | $\sim \ln N$ 👍 | $\sim \ln N$ 👍 | $\sim \ln N / \ln\ln N$ 👍 |
| $C$ | const 👍 | $\langle k \rangle / N$ 👎 | const 👍 | const. 👍 |
| $p_k$ | $\delta(k, k_C)$ 👎 | Poisson 👎 | shifted Poisson 👎 | $\sim k^{-\gamma}$ 👍 |

It is worth concentrating on the mechanisms!

# Further network growth models

There is no strict universality: empirical exponents are system dependent and not 3!




Can we produce different exponents with slight modification of BA, keeping the concept?

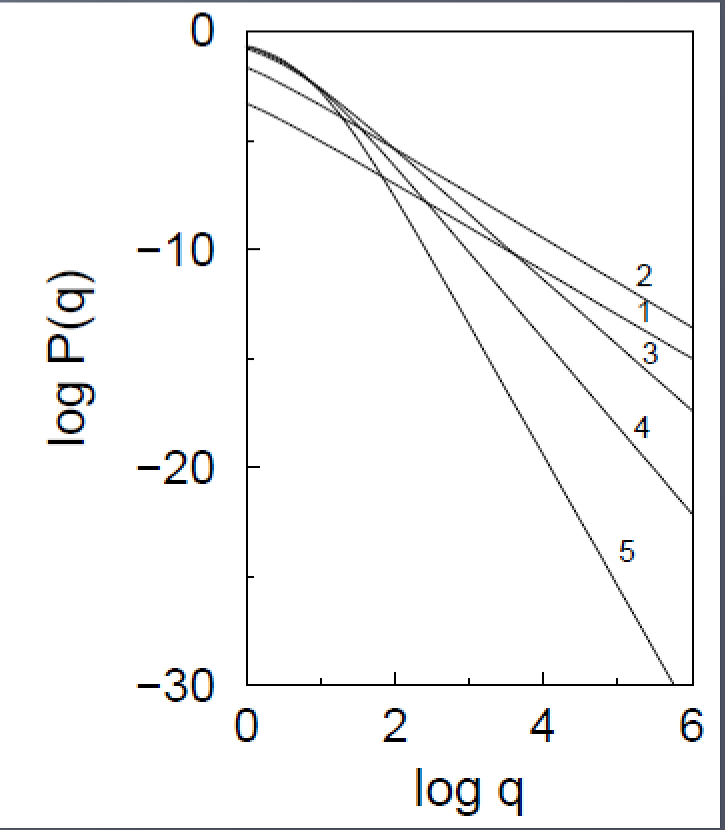
# Further network growth models

Yes!

Initial attractiveness : $\Pi(k) \sim A+(k\text{-}m)=A+q$

$\rightarrow$ P(k) ~ $k^{-\gamma}$ where $\gamma$=2 + A/m

Tunable exponent between 2 and ∞




A/m=0.001; 0.05; 1; 2; 4

Dorogovtsev, Mendes, Samukhin, Phys. Rev. Lett. 85, 4633 (2000)

# We saw *k*(*t*)

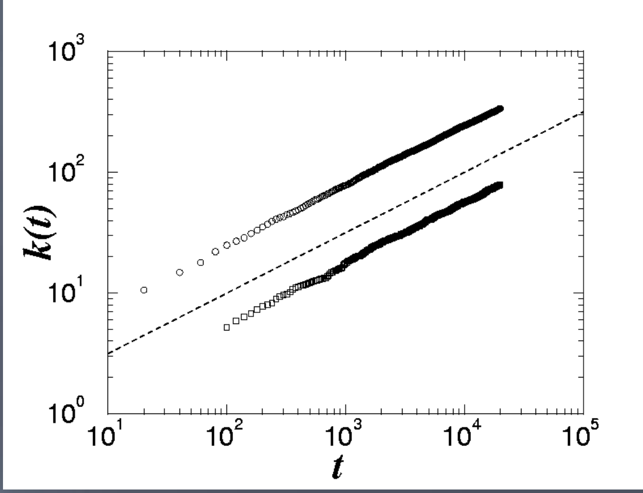
Let us describe the time evolution of degrees

$$\frac{\partial k_i}{\partial t} \propto \Pi(k_i) = A\frac{k_i}{\sum_j k_j} = \frac{k_i}{2t}$$

During a unit time (time step):

$\Delta k = m \;\rightarrow A = m$

$$k_i(t) = m\sqrt{\frac{t}{t_i}} \sim t^{\beta} \quad \text{with } \beta = 1/2$$




This tells us that what matters is the age of the nodes. Old nodes will always have advantage!

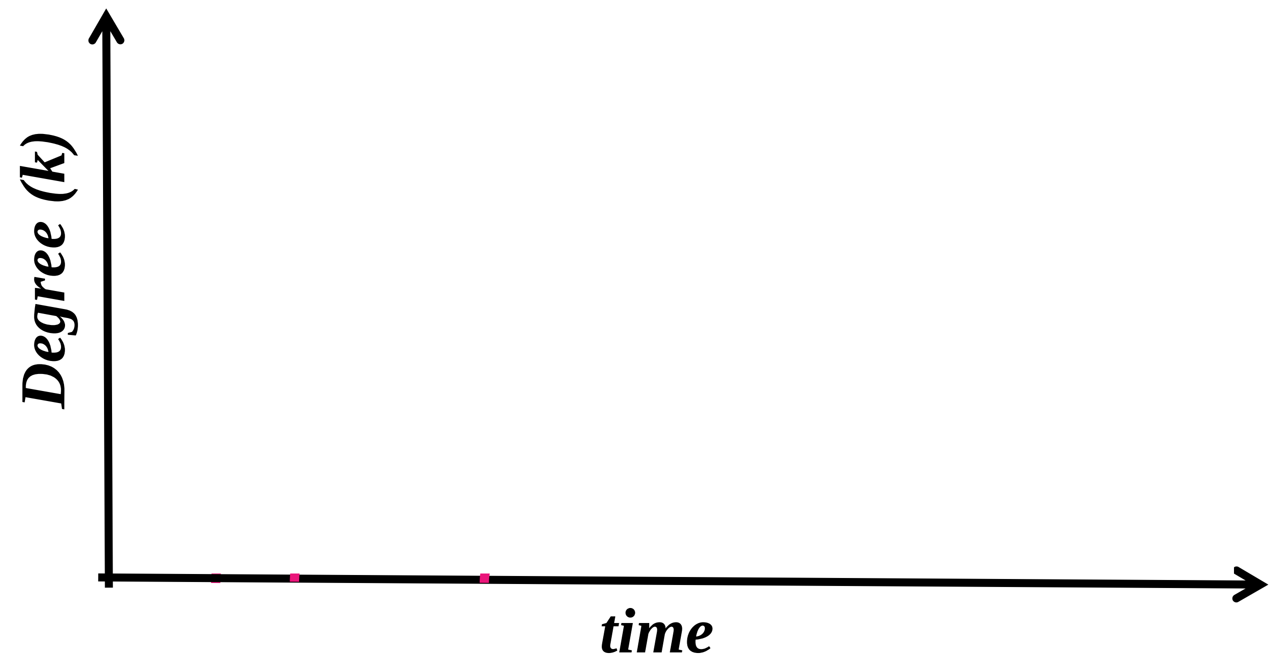
Further question: Can new ones make it? (Google!)

A.-L.Barabási, R. Albert and H. Jeong, *Physica* A **272,** 173 (1999)

# Further network growth models

## Fitness Model: Can Latecomers Make It?

BA model: $k(t) \sim t^{1/2}$ (first mover advantage)



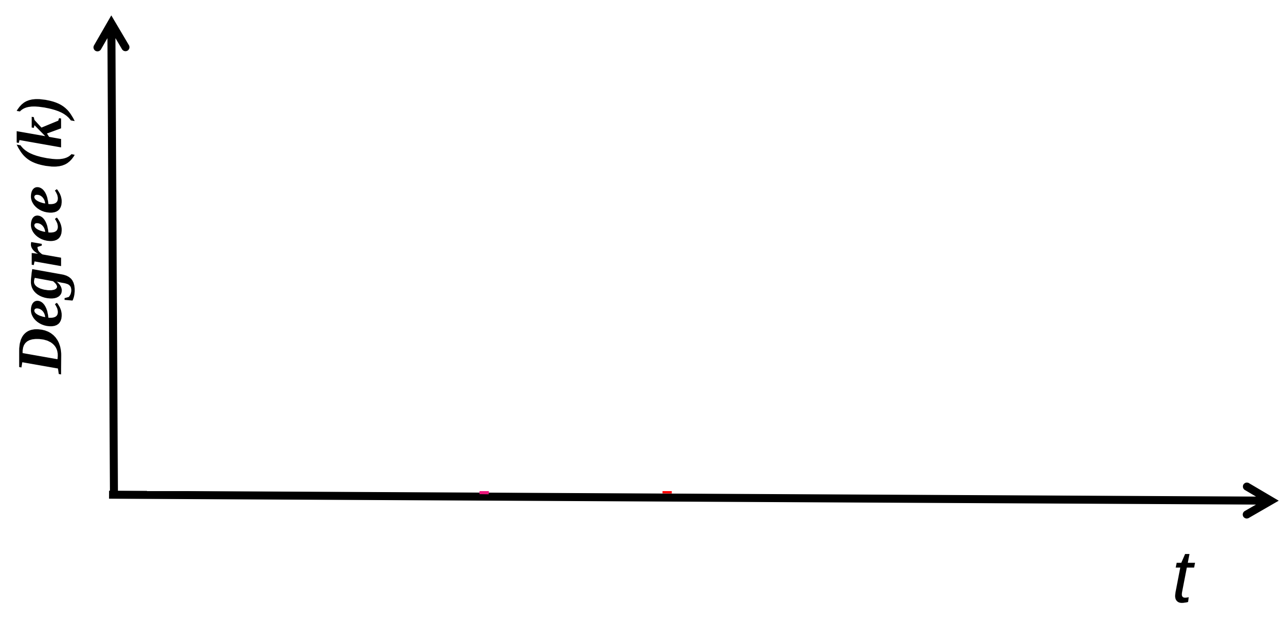
# Further network growth models

**Fitness model:** fitness $(\eta)$

$k(\eta,t) \sim t^{\beta(\eta)}$

$$\Pi(k_i) = \frac{\eta_i k_i}{\sum_j \eta_j k_j}$$

$\beta(\eta) = \eta / C$



# Preferential attachment

A key element in the BA model (and related ones) is that the new nodes attach to the old ones via preferential attachment.

We have seen that this is quite common assumption – but it is strange!

Imagine, you are making a new www site and put appropriate links into it. Do you have to search through the $10^{10}$ web sites and make a statistics to calculate the probability of linking to one of them?!

No way!

# Preferential attachment

In reality we cannot test a huge network. The decision is local – but what comes out *is* preferential attachment. (Self organization – invisible hand…)

There could be other mechanisms. It is a legitimate question to ask: Does a system obey PA?

We can measure this!

# Preferential attachment

$$\frac{\partial k_i}{\partial t} \propto \Pi(k_i) \sim \frac{\Delta k_i}{\Delta t}$$

Plot the change in the degree $\Delta k$ during a fixed time $\Delta t$ for nodes with degree $k$, and you get ~ $\Pi(k)$

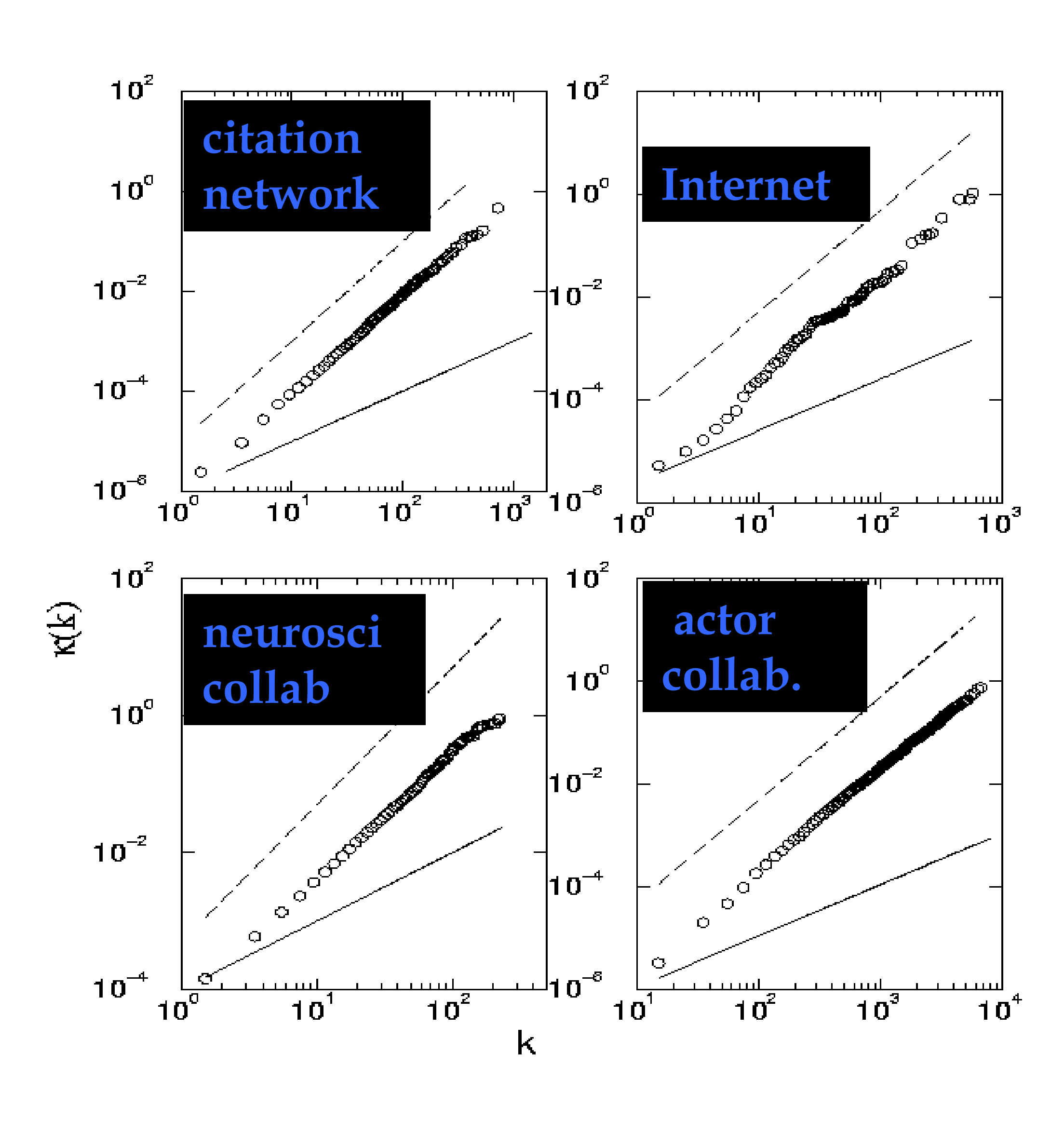
To reduce noise, plot the sum of *Π(k)* over *k*:

$$\Pi_<(k) = \sum_{k'<k} \Pi(k')$$

One has to measure the cumulative increment of the nodes with degree smaller than *k*

(Jeong, Neda, A.-L. B, Europhys Letter 2003; cond-mat/0104131)

# Preferential attachment




Plots of the cumulative

$$\Pi_{<}(k) = \sum_{k'<k} \Pi(k')$$

**No pref. attach:**
$\kappa \sim k$

**Linear pref. attach:**
$\kappa \sim k^2$

# Preferential attachment

What could be the local mechanism leading the PA?

1. Copying mechanism
   directed network
   select a node and an edge of this node
   attach to the endpoint of this edge

2. Walking on a network
   directed network
   the new node connects to a node, then to every
   first, second, … neighbor of this node

3. Attaching to edges
   select an edge
   attach to both endpoints of this edge (clustering!)

4. Node duplication
   duplicate a node with all its edges
   randomly prune edges of new node

# Vertex copying mechanism

Consider the citation network.
Nodes: Papers
Directed: Out degrees - references cited
In degrees – citing papers

In an ideal case those papers are cited, which have been read by the authors and have impact on the results.

Authors are often sloppy: They simply copy the reference list of other papers. (This is evidenced by propagation of typos.)

Similar mechanism may work in other networks too.

# Vertex copying mechanism

To make it more realistic: Only a fraction of an old paper's refs is copied and the remaining is filled in with others selected at random. (For simplicity we assume that all bibliographies have the same size, c.)

| Old | | New |
| --- | --- | --- |
| 1) aaa | prob. $\gamma$ → | 1) aaa (copied) |
| 2) bbb | prob. $1-\gamma$ → | 2) bla (randomly chosen) |
| c) zzz | → | c) zzz |

Kleinberg et al. 1999

# Vertex copying mechanism

As a result, we will have on the average $c\gamma$ copied items and $c(1 - \gamma)$ ones selected at random.

Starting set: E.g., $n_0$ vertices interconnected randomly such that everybody has *c* out links (no mulitlinks allowed).

Q: What is the in degree distribution?

Node *i* can get a link in two ways: a) part of a list and copied b) randomly selected. The number of nodes at time *t* is *n.*

a) Node *i* has indegree $k^{in}_i$. The prob. of choosing *i* is thus $\gamma k^{in}_i / n$. (Case a).
b) $c(1 - \gamma) / n$.

# Vertex copying mechanism

$$\frac{\gamma {k_i}^{in}}{n} + \frac{(1-\gamma)c}{n} = \frac{\gamma {k_i}^{in} + (1-\gamma)c}{n}$$

The expected number of nodes with in degree *k* receiving a new link is

$$np_k \frac{\gamma k + (1-\gamma)c}{n} =$$

$$p_k[\gamma k + (1-\gamma)c] = \frac{c(k+a)}{c+a} p_k$$

with $a = c(\frac{1}{\gamma} - 1)$

which can be rewritten as

resulting in a rate equation:

$$(n+1)p_k(n+1) = np_k(n) + \frac{c(k-1+a)}{c+a} p_{k-1}(n) - \frac{c(k+a)}{c+a} p_k(n)$$

$$(n+1)p_k(n+1) =$$
$$np_k(n) + \frac{c(k-1+a)}{c+a}p_{k-1}(n) - \frac{c(k+a)}{c+a}p_k(n)$$

$$(n+1)p_0(n+1) = np_0(n) + 1 - \frac{ca}{c+a}p_0(n)$$

*k*=0

For *n*→oo stationary solution:

$$p_k = \frac{c}{c+a}[(k-1+a)p_{k-1}(n) - (k+a)p_k(n)]$$
$$p_0 = 1 - \frac{ca}{c+a}p_0$$

*k*=0

$$p_0 = \frac{1+a/c}{a+1+a/c}$$

For every *k* iterative sol'n. The asymptotics is: $p_k \sim k^{-\alpha}$

with $\alpha = 2 + a/c = 1 + 1/\gamma$

Tunable exponent

For more details see Newman's book

# Summary of models

| | | |
|---|---|---|
| Linear growth, linear pref. attachment | $\gamma=3$ | Barabási and Albert, 1999 |
| Nonlinear preferential attachment $\Pi(k_i)\sim k_i^{\alpha}$ | no scaling for $\alpha\neq1$ | Krapivsky, Redner, and Leyvraz, 2000 |
| Asymptotically linear pref. attachment $\Pi(k_i)\sim a_\infty k_i$ as $k_i\to\infty$ | $\gamma\to2$ if $a_\infty\to\infty$ <br> $\gamma\to\infty$ if $a_\infty\to0$ | Krapivsky, Redner, and Leyvraz, 2000 |
| Initial attractiveness $\Pi(k_i)\sim A+k_i$ | $\gamma=2$ if $A=0$ <br> $\gamma\to\infty$ if $A\to\infty$ | Dorogovtsev, Mendes, and Samukhin, 2000a, 2000b |
| Accelerating growth $\langle k\rangle\sim t^{\theta}$ constant initial attractiveness | $\gamma=1.5$ if $\theta\to1$ <br> $\gamma\to2$ if $\theta\to0$ | Dorogovtsev and Mendes, 2001a |
| Internal edges with probab. $p$ <br> Rewiring of edges with probab. $q$ | $\gamma=2$ if $q=\frac{1-p+m}{1+2m}$ <br> $\gamma\to\infty$ if $p,q,m\to0$ | Albert and Barabási, 2000 |
| $c$ internal edges or removal of $c$ edges | $\gamma\to2$ if $c\to\infty$ <br> $\gamma\to\infty$ if $c\to-1$ | Dorogovtsev and Mendes, 2000c |
| Gradual aging $\Pi(k_i)\sim k_i(t-t_i)^{-\nu}$ | $\gamma\to2$ if $\nu\to-\infty$ <br> $\gamma\to\infty$ if $\nu\to1$ | Dorogovtsev and Mendes, 2000b |
| Multiplicative node fitness $\Pi_i\sim\eta_i k_i$ | $P(k)\sim\frac{k^{-1-C}}{\ln(k)}$ | Bianconi and Barabási, 2001a |
| Edge inheritance | $P(k_{in})=\frac{d}{k_{in}^{\sqrt{2}}}\ln(ak_{in})$ | Dorogovtsev, Mendes, and Samukhin, 2000c |
| Copying with probab. $p$ | $\gamma=(2-p)/(1-p)$ | Kumar *et al.*, 2000a, 2000b |
| Redirection with probab. $r$ | $\gamma=1+1/r$ | Krapivsky and Redner, 2001 |
| Walking with probab. $p$ | $\gamma\simeq2$ for $p>p_c$ | Vázquez, 2000 |
| Attaching to edges | $\gamma=3$ | Dorogovtsev, Mendes, and Samukhin, 2001a |
| $p$ directed internal edges $\Pi(k_i,k_j)\propto(k_i^{in}+\lambda)(k_j^{out}+\mu)$ | $\gamma_{in}=2+p\lambda$ <br> $\gamma_{out}=1+(1-p)^{-1}+\mu p/(1-p)$ | Krapivsky, Rodgers, and Redner, 2001 |

# Homework

Generate graphs with nonlinear preferential attachment:

$$\Pi(i) = \frac{k_i^\beta}{\sum_j k_j^\beta}$$

Calculate the degree distribution and try to find a characteristic degree as a function of $\beta$. (Chose $\beta$ values not far from 1.)